

RIBBONLENGTH OF FOLDED RIBBON LINKS

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ABSTRACT. Given a thin strip of paper, tie a knot, connect the ends, and flatten into the plane. This is a physical model of a folded ribbon knot in the plane, first introduced by Louis Kauffman. We study the folded ribbonlength of these folded ribbon knots, which is defined as the knot's length-to-width ratio. By finding new methods of creating folded ribbon knots, we improve upon existing upper bounds for the folded ribbonlength of $(2, q)$ torus links, twist knots, and pretzel links. For example, we find the folded ribbonlength of any twist knot T_n to be given by $\text{Rib}([T_n]) \leq n + 6$. Applying this to the figure-eight knot yields a ribbonlength of 8. We use accordian, or crinkle, folds to construct knots where the ribbonlength remains constant as its crossing number increases. We apply these results to $T_{2,q}$ torus knots for q odd and T_n twist knots for n odd. For example, we find the ribbonlength of $T_{2,q}$ torus knots with q odd to be $\text{Rib}([T_{2,q}]) \leq 16$. We use this construction to prove that $\alpha = 0$ for the ribbonlength-crossing number conjecture: $c_1 \cdot (\text{Cr}(K))^\alpha \leq \text{Rib}([K])$.

1. KNOTS AND LINKS

We will begin by introducing some of the basic definitions and concepts in knot theory. These definitions can be found in any introductory knot textbook, for example: [1, 5].

A *knot* is a closed curve in space that does not intersect itself. A *link* is a disjoint collection of one or more of these knots, each of which is considered to be a *component* of said link. Two links are *equivalent* if you are able to move from one to the other through a series of deformations that do not cause the link to cross through itself. A *link diagram* is a projection of a link to the plane. At each crossing of a link diagram, the piece of the link that passes under the other is shown to have a small gap. The *crossing number* of a link is the least number of crossings in any link diagram out of all possible diagrams for that link. One of the simplest links is the Hopf link, which consists of two loops that are linked together exactly once (see Figure 1, left). Shown on the right of Figure 1 is a trivial link of two components, also called an unlink.

In order to study folded ribbon links, we start by focusing on specific types of links called *polygonal links*. A polygonal link is a link that is made of a finite number of edges and vertices. We are often interested in the *stick number* of polygonal links, which is the number of edges,

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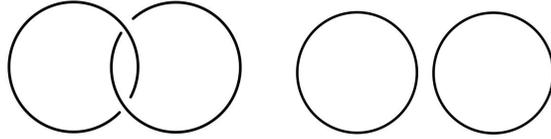


FIGURE 1. The Hopf link (left) has a crossing number of two, while a trivial link of two components (right) has a crossing number of zero.

or “sticks,” used to construct a certain link. In order for the diagram of a polygonal link to be considered *regular*, it must be true that no more than two points on the link project to any given point on the link diagram, and that no vertex projects to the same point as any other point on the link. An example of a regular polygonal link diagram can be seen in Figure 2.

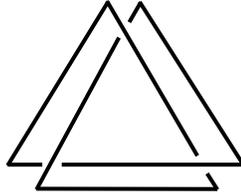


FIGURE 2. The trefoil knot can be shown as a polygonal link diagram of six sticks.

Consider a link with two components, J and K . When we draw the link as an oriented diagram, each crossing can be given a sign $+1$ or -1 , as shown in Figure 3. To find the *linking number*, add up the signs of all the crossings between the two components J and K and divide this sum by 2. Formally, the linking number $\text{Lk}(J, K)$ is given by:

$$\text{Lk}(J, K) = \frac{1}{2} \sum_{\text{crossings}} \text{sign}(c),$$

where $\text{sign}(c)$ is the sign of each crossing. For instance, the Hopf link in Figure 1 has linking number ± 1 depending on the orientations chosen. It is known that the linking number is an integer, and the linking number is a link invariant.

1.1. Torus Links. Torus links are equivalent to curves that wrap around the torus without self intersection (see leftmost image in Figure 4). We denote torus links by $T_{p,q}$, with p corresponding to the number of wraps around the long way of the circumference of the torus, and q corresponding to the number of wraps around the short way. In a certain \mathbb{R}^2 planar projection of a torus link, p



FIGURE 3. The crossing on the left is -1 ; the crossing on the right is $+1$.

corresponds with the amount of strands that travel in circular arcs around a central region, and q corresponds with the certain “twists” created with each wrap around the short side of the torus that can be congregated in one side of the link for better analysis. This projection can be seen in the right image in Figure 4.

There are several well-known facts about torus links that can be found in many foundational knot theory textbooks [1, 5]. For example, torus links of $T_{p,q}$ are equivalent to $T_{q,p}$. The greatest common factor of p and q corresponds to the number of components in the link, meaning torus links of one component, or torus knots, are created when p and q only have 1 as a common factor. The middle image in Figure 4 shows a $T_{2,2}$ torus link, which has two components. The crossing number of a torus link is the $\min\{((p-1) \cdot q), (p \cdot (q-1))\}$. Looking specifically at $T_{2,q}$ torus links, we see that the crossing number is $\text{Cr}(T_{2,q}) = q$. For $T_{2,q}$ torus links where q is even, $T_{2,q}$ is a two component link with linking number $\text{Lk}(T_{2,q}) = \pm q/2$. Note that the trefoil knot is a torus knot $T_{2,3}$ or $T_{3,2}$.

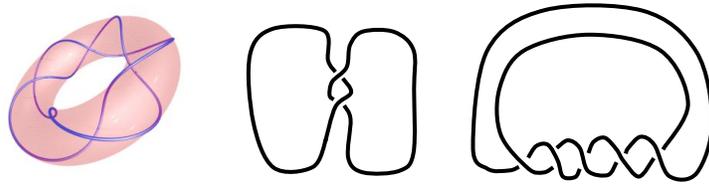


FIGURE 4. Left image shows $T_{5,2}$ on torus. Center image shows $T_{2,2}$ with two half-twists. Right image shows $T_{2,5}$ with 5 half-twists.

Definition 1 (Half-twist). Take two strands of a knot. Passing one strand over the other and creating one crossing is one *half-twist*, as shown in the first two images in Figure 5. Passing one strand to the opposite side, and then passing it back to the original side, with alternating crossings, creates two half-twists, or one *full-twist* (shown in the center image in Figure 5). These alternating crossings are positive when all over-strands have positive slopes, and are negative when the over-strands have negative slopes. This distinction is shown in the rightmost pictures in Figure 5 as the left image of the two shows n positive twists, and the rightmost shows n negative twists. Note that the way we define positive and negative half-twists may differ in other literature.

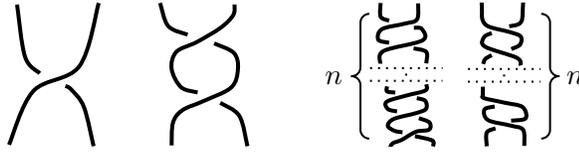


FIGURE 5. Left and center images show the construction of one and two half-twists. Right image shows n positive twists on the left and n negative twists on the right.

Looking specifically at torus links of $T_{2,q}$, we see that they are built by q half-twists. This construction can be seen in Figure 4 in the rightmost image. We see how the number of wraps the short way around the torus manifest q half-twists.

1.2. **Twist Knots.** A *twist knot* with n half-twists, denoted as T_n , can be created by adding n half-twists to two strands, then hooking the ends of the strands together to form a closed loop. This is also known as the clasp region (see Figure 6).

A twist knot is an *alternating knot*, meaning it has a projection where the crossings alternate between over and under as we travel around the knot diagram. Figure 6 shows T_1 twist knot on the left, which is also the trefoil knot. The figure eight knot is a twist knot T_2 .

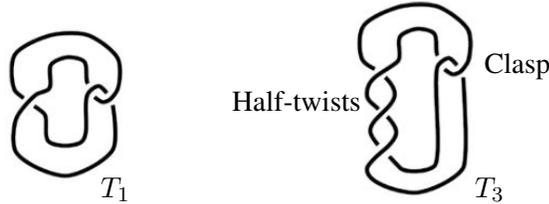


FIGURE 6. On the left, T_1 twist knot; on the right, T_3 twist knot.

1.3. **Pretzel Links.** A 3-strand pretzel link $P_{p,q,r}$ is made of three groupings of p , q , and r half-twists, called *strands*. When these strands are arranged in the same fashion as the groupings of half-twists shown in Figure 5, it can be seen that each strand has four ends: top left, top right, bottom left, and bottom right. With strands p , q , and r laid out from left to right, we begin to create a pretzel link by joining the top right and bottom right of strand p to the top left and bottom left of strand q , respectively. In a similar fashion, we connect the right ends of strand q to the left ends of strand r . Finally, we connect the top right and bottom right ends of strand r to the top left and bottom left ends of strand p , respectively. This joining method can be seen in Figure 7.

The crossing number and the number of components of a $P_{p,q,r}$ pretzel link both depend on the sign and parity of p , q , and r . For example, when p , q , and r are all odd and have the same sign, the knot diagram is reduced and alternating. This means that the crossing number of $P_{p,q,r}$ is equal to

$|p| + |q| + |r|$. We know that if zero or one strands have an even number of half-twists, then $P_{p,q,r}$ is a link of one component. If two strands have an even number of half-twists, then $P_{p,q,r}$ is a link of two components, and if all three strands have an even number of half-twists, then $P_{p,q,r}$ is a link of three components. Additionally, any twist knot T_n can be constructed as a 3-strand pretzel link $P_{n,1,1}$.

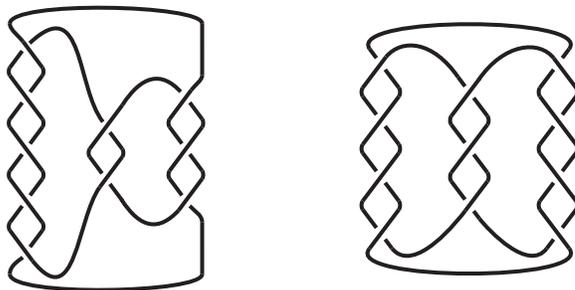


FIGURE 7. Pretzel link $P_{-5,2,3}$ (left) is a link of one component, while pretzel link $P_{4,3,-4}$ (right) is a link of two components.

The definition of 3-strand pretzel links can also be generalized to define n -strand pretzel links, made with (p_1, p_2, \dots, p_n) half-twists.

2. FOLDED RIBBONS

2.1. Constructing Folded Ribbon Links. Intuitively, we can imagine a folded ribbon knot to be a rectangular strip of paper that we tie into a knot, connect the ends, and flatten into the plane. More formally, we create a folded ribbon link by taking a polygonal link diagram L and placing two parallel lines at an equal distance from each original edge, with one on each side. This results in a ribbon with width w . At each vertex of the original diagram, we modify the joining of the edges to create folds. Each of these folds includes a fold line, which acts like a mirror that reflects its adjacent edges into each other. This results in a folded ribbon link, denoted as L_w . A comparison of a polygonal knot and its corresponding folded ribbon knot can be seen in Figure 8. Observe that, while a minimum of six edges is required to construct a trefoil knot in space, the trefoil can be constructed as a folded ribbon knot with only five sticks. The least number of sticks required to construct a folded ribbon link will always be less than or equal to the least number of sticks required to construct an equivalent link in space.

We create an *oriented* folded ribbon link by assigning a direction of travel along the knot diagram. Once a folded ribbon link is oriented, we can classify each of its folds as either an *underfold* or an *overfold*. When following the direction of orientation, an underfold is encountered when the first piece of ribbon traversed is over of the second piece. Similarly, an overfold is encountered

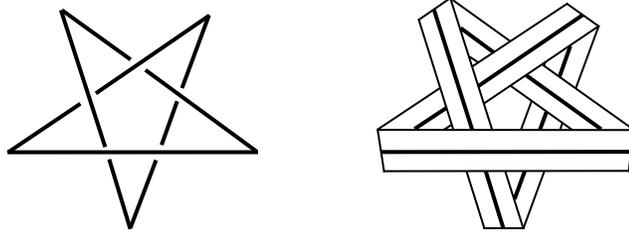


FIGURE 8. We create a folded ribbon trefoil knot (right) by adding width and folds to a polygonal trefoil knot diagram (left).

when the first piece traversed is under the second piece. This distinction can be seen in Figure 9, where we see two folds at angle $\pi/2$. Folds at this angle result in two overlapping right isosceles triangles, which is further discussed in Remark 3 and will be an important observation through the rest of this report. We also classify folds by the direction that the second ribbon piece is traveling in relation to the first piece. This is primarily split into left and right folds, but also includes folds at angle π , which are neither left folds nor right folds and cause the second piece of ribbon to be antiparallel to the first piece. A more detailed description of how these folds behave, as well as a more formal definition of folded ribbon knots and links in general, can be found in [3].

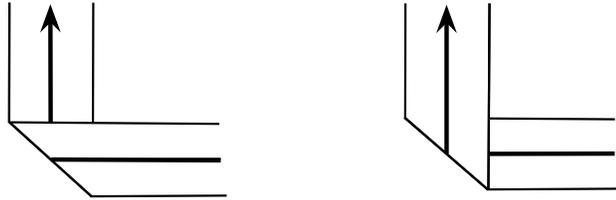


FIGURE 9. The distinction between an underfold (left) and an overfold (right) depends on the orientation of the link.

2.2. Folded Ribbonlength.

Definition 2. The ribbonlength of a folded ribbon link L_w is defined as the ratio of the length of the link L to its width w :

$$\text{Rib}(L_w) = \frac{\text{Length}(L)}{w}.$$

The ribbonlength problem seeks to find the shortest piece of ribbon needed to tie a folded ribbon knot for a particular knot or link type. This is denoted $\text{Rib}([L])$, and is the infimum of the ribbonlength over all possible configurations of the link L :

$$\text{Rib}([L]) = \inf_{L_w \in [L]} \text{Rib}(L_w).$$

Using infimum is necessary because the ribbonlength of the unknot 0_1 is 0. There exists a polygonal unknot diagram that can be reduced to a 2-stick unknot, and we can make the width of such a diagram arbitrarily large. This highlights the importance of the infimum in the ribbonlength definition, as it allows for considering the smallest possible length configuration of the knot.

Remark 3. Observe that the ribbonlength of any square ribbon segment is 1. Additionally, we can fold a square ribbon segment at angle $\pi/2$ to make two right isosceles triangles, each with a ribbonlength of $1/2$, accounting for a total ribbonlength of 1. This remark will be essential to many of the folded ribbon link constructions that we describe in our report.

Previous research has managed to lower the ribbonlength upper bound for constructing many knot families, for example, torus knots, twist knots, and pretzel links. Kauffman [6] has proved that the trefoil knot has $\text{Rib}([3_1]) \leq 5 \cot(\frac{\pi}{5}) \leq 6.89$, folding the trefoil in a pentagon shape. Denne et al. [3] have discovered two different ways of constructing the trefoil knot such that $\text{Rib}([3_1]) \leq 6$. Kauffman has proved that the figure eight knot has $\text{Rib}([4_1]) \leq 10.3$, while Denne et al. found a construction such that $\text{Rib}([4_1]) \leq 10$. In our paper, we aim to lower the ribbonlength upper bound further.

2.3. Folded Ribbonlength and Crossing Number. The relationship between the ribbonlength of a knot K and its crossing number $\text{Cr}(K)$ is an open and interesting question in the study of folded ribbon knots. The primary objective is to establish constants c_1 , c_2 , α , and β such that the following inequality holds:

$$c_1 \cdot (\text{Cr}(K))^\alpha \leq \text{Rib}([K]) \leq c_2 \cdot (\text{Cr}(K))^\beta.$$

Diao and Kusner [3] conjectured that $\alpha = \frac{1}{2}$ and $\beta = 1$. Denne [2] has proven that $\beta = \frac{3}{2}$ for all knots, which indicates that the upper bound of the folded ribbonlength grows faster than linearly with respect to the crossing number. In addition, $\beta = 1$ is known for many knot families [3]. The discrepancy between the general case and specific families underscores the diversity in knot behavior and the necessity for detailed studies of individual knot types.

Our research is focused on lowering the upper bound constant c_2 for some knot families by discovering new folding methods. Efficiently constructing certain knots allows us to establish practical upper bounds for their ribbonlength, especially when $\beta = 1$. By reducing the value of c_2 , we aim to tighten the bounds and provide more precise information about the folded ribbonlength of various knots and links. Additionally, in Section 6, we have proved that $\alpha = 0$ for certain types of knots. This remarkable result implies that for these particular knots, the folded ribbonlength can be bounded below by a constant, regardless of the crossing number. This finding contrasts with the general behavior suggested by Diao and Kusner's conjecture and offers new aspects of studying ribbonlength.

3. THE SPRING METHOD

In the following sections, please prepare several strips of paper with constant width so that you can build all the constructions discussed later. Folding actual paper ribbons will be helpful for understanding.

First, we will detail the *spring method* construction for half-twists. This construction has already been detailed in [3] and applied to torus knots of $T_{2,q}$ where q is odd. Denne et al. found a

folded torus ribbon knot with ribbonlength of $2q$ using this construction, with the number of sticks used $2q + 2$. We will apply this construction to torus links of $T_{2,q}$, where q is even, and find the ribbonlength of these links.

Construction 4 (Spring Method). Take two separate pieces of ribbon, AB and CD . Cross the two strands in an X formation such that AB crosses over CD (see Figure 10, second image). This creates one half-twist, as it creates one crossing as one strand crosses to the other side, as seen in Figure 5. Fold ribbon D over AB toward C , then fold ribbon B over D toward A (Figure 10 images 3 & 4). These two folds create one crossing, corresponding to another half-twist. Looking back at Figure 5 we can see that two half-twists have two crossings and that each strand should exit the crossing on the same side as they entered. The double fold over creates one crossing and Figure 10 shows strands overlapping themselves. To continue creating half-twists, fold ribbon D over B then ribbon B over D . The pieces of ribbon are folded over in pairs to create one half-twist as both ends must fold over each other to create a new crossing. Continue to fold ribbon D over B , then ribbon B over D until the desired amount of half-twists has been created. If there are an even number of half-twists, end D will be folded toward end C , and end B toward end A ; if there is an odd number of half-twists, end D will be folded away from end C and end B away from end A . This method will be referred to as the *spring method*.

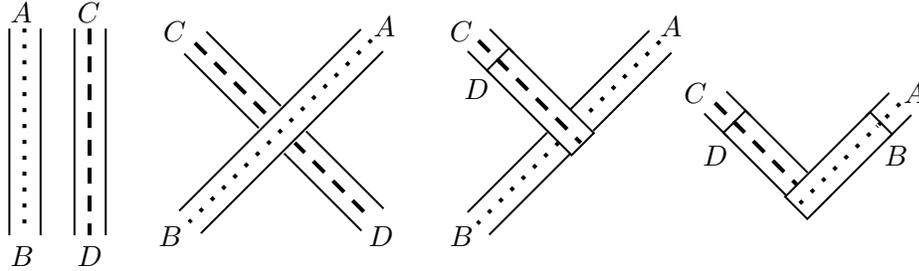


FIGURE 10. Construction of spring method.

Construction 5 ($T_{2,q}$ spring). To construct a torus link of $T_{2,q}$, where q is even, start by building q half-twists using the spring method. The ribbon B should be on top of A and D should be on top of C , illustrated by the second image in Figure 11. As shown by the leftmost image in Figure 11, to make a $T_{2,q}$ link, where q is even, we must connect ribbon A to B and ribbon C to D . Therefore, take the ends of A and B and join them. Now shrink the extra length until it is against the square region of all the folds. Then join the ends of C and D together and trim them until they also rest against the boundary of the square region.

Theorem 6. For any $T_{2,q}$ torus link where q is even, the folded ribbonlength is $\text{Rib}([T_{2,q}]) \leq 2q$.

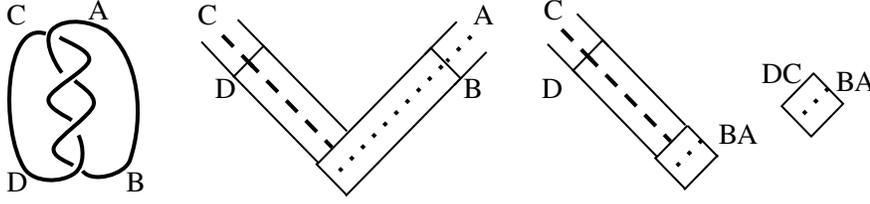


FIGURE 11. Construction of connection for $T_{2,q}$ torus knots where q is even.

Proof. In Construction 5, we have constructed a $T_{2,q}$ torus link. We now show that the ribbonlength for this torus link is $2q$. Following Construction 5, we have constructed q half-twists using the spring method. Each of these half-twists consists of two square lengths of ribbon, and each of these squares has ribbonlength 1 by Remark 3. The beginnings and ends of each link component meet up without the use of any additional ribbonlength. Therefore the total ribbonlength required for this construction is $2q$. \square

Corollary 7. *The number of sticks required for a spring construction of a $T_{2,q}$ torus link, where q is even, is equal to $2q$.*

Proof. Following Construction 4, we can see that each half-twist is composed of two squares, which correspond to one stick. While joining the ends of the link, no sticks are added. Therefore, the number of sticks is equal to $2q$. \square

Corollary 8. *For any torus links $T_{2,q}$, $\text{Rib}([T_{2,q}]) \leq 2\text{Cr}(T_{2,q})$*

Proof. As the crossing number of a $T_{2,q}$ torus link is q , the ribbonlength of a $T_{2,q}$ torus link is $\text{Rib}(T_{2,q}) \leq 2\text{Cr}(T_{2,q})$. \square

Corollary 9. *For any two component links L , with linking number $|\text{Lk}(L)| = n$, the folded ribbonlength is $\text{Rib}([L]) \leq 4n$.*

Proof. For all two components links $L = J \cup K$, with linking number $|\text{Lk}(L)| = n$, there exists a $T_{2,2n}$ torus link with $\text{Lk}(T_{2,2n}) = n$ by Section 1.1. Based on Theorem 6, the ribbonlength for $T_{2,2n}$ satisfies $\text{Rib}(T_{2,2n}) \leq 4n$. Therefore, for any link with linking number n , we have $\text{Rib}(L) \leq 4n$. \square

4. THE WRAP METHOD

In this section, we will detail a new construction method, the *wrap method*, which helps optimize the ribbonlength of knots that consist of half-twists. We will apply the wrap method to $T_{2,q}$ torus links, twist knots, and pretzel links.

Construction 10 (Wrap Method). Begin with two separate pieces of ribbon, AB and CD . Place ribbon DC horizontally below and ribbon AB vertically on the top, as shown by the first picture

in Figure 12. Fold ribbon A towards C with a $\pi/2$ fold so that ribbon A lies on top of CD , and the ribbon B is pointing downward, illustrated by the second image in Figure 12. Wrap ribbon B under CD such that the ribbon B now points upward at the back (third picture); this is the first half-twist we created using the wrap method. Fold ribbon B downwards over ribbon CD and A to create the second half-twist, as shown in the right-most image in Figure 12. By repeating this process of wrapping ribbon B around ribbon CD and A at the wrap region, we are able to construct as many half-twists as we want. This method will be referred to as the *wrap method*.

Based on the construction of the first two half-twists using the wrap method, we observe that for an odd number of half-twists, ribbon B will be pointing upwards at the back, while for an even number of half-twists, ribbon B will be pointing downwards at the front. The half-twists created following this construction are negative half-twists based on Definition 1. In order to create positive half-twists, fold ribbon A towards ribbon D instead of C (comparing to the second image in Figure 12), and then start the wrap similarly.

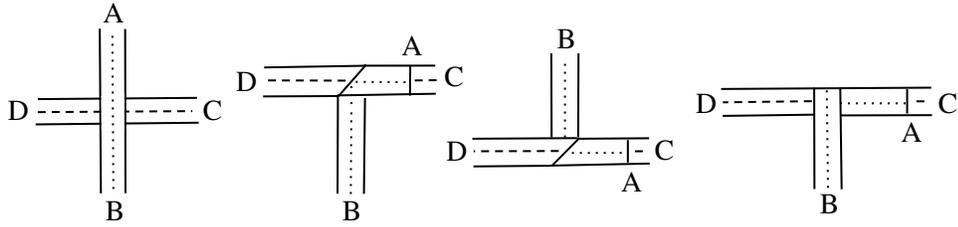


FIGURE 12. Construction of the wrap method of twisting.

Lemma 11. *Applying the wrap method, n half-twists can be constructed using at least $n + 2$ units of ribbonlength and $n + 3$ number of sticks.*

Proof. Following the Construction 10 and Figure 12, the half-twists created by the wrap method are concentrated in the wrap area which is a square region. The ribbonlength measured is the amount of ribbon folded in the square region, excluding the extra length of the ends of the ribbon. Looking at the second picture in Figure 12, the $\pi/2$ fold of ribbon A towards C is 1 unit of ribbonlength based on Remark 3. Folding ribbon B upwards behind CD uses another 1 unit. The part of CD (a square) that is wrapped by ribbon B is another unit of ribbonlength. Therefore, constructing the first half-twist using the wrap method needs 3 units of ribbonlength. Each subsequent half-twist uses only 1 unit of ribbonlength as each wrap of B above or below CD takes 1 square fold. Hence, creating n half-twists uses $n + 2$ units of ribbonlength. For the number of sticks, each square fold contributes to 1 stick, and the $\pi/2$ fold contributes to 2 sticks. Therefore, creating the first half-twist requires 4 sticks, and each of the following half-twist just needs 1 more sticks. Hence, we need $n + 3$ sticks to construct n half-twists using the wrap method. \square

Remark 12. Figure 13 shows two link diagrams demonstrating the odd and even number of half-twists constructed following the wrap method in Construction 10. As any torus link of $T_{2,q}$ can

be built with q half-twists, we now need to consider how to join the four ends so we can create a torus link. When q is odd, the torus link only has one component. As shown by the left picture in Figure 13, D needs to connect with A , and B has to connect with C . For q is even, the torus link of $T_{2,q}$ has two components. Therefore, B connects to A , and C connects to D , shown by the right picture in Figure 13.

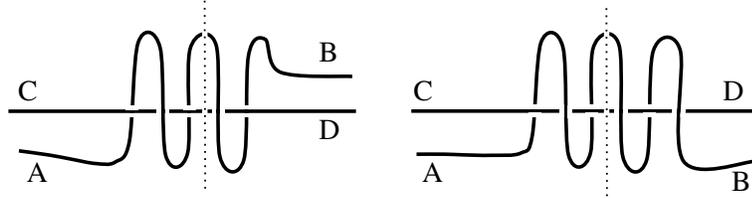


FIGURE 13. Link diagram of half-twists. Left shows odd number of half-twists, right shows even number of half-twists.

Construction 13 ($T_{2,q}$ odd wrap). To construct a torus knot of $T_{2,q}$, where q is odd, start with constructing with an odd number of half-twists applying the wrap method from Construction 10. We observe that ribbon B is underneath ribbon CD and pointing upward, as shown in the first picture in Figure 14.

In order to construct a $T(2, q)$ torus knot, ribbon D should be connected to A and ribbon B should be connected to C , which is detailed explained by Remark 12 and the left picture in Figure 13.

To do so, fold ribbon A towards ribbon D so that ribbon A is on top of D , as shown in the top-middle image of Figure 14). Now connect ribbon A and D and trim the join until it reaches the wrap region of ribbon B , demonstrated by the top-right picture in Figure 14. Then flip over the knot such that ribbon B is over top of all other strands, shown in the left-bottom image. Fold ribbon B with the $\pi/2$ angle towards C and then join them and trim until the join is against the wraps of B , illustrated by the last two images of Figure 14.

Construction 14 ($T_{2,q}$ even wrap). To construct a torus link of $T_{2,q}$, where q is even, applying the wrap method with an even number of half-twists following Construction 10. As shown by the first picture in Figure 15, ribbon B is on top of ribbon CD and A and pointing downward.

In order to build $T(2, q)$ torus link, ribbon C should be joined with D and ribbon A joined with B with an even q , explained by Remark 12 and the right picture in Figure 13.

Fold ribbon B towards ribbon A and C with angle $\pi/2$ so that ribbon B is on top, shown by the second image in Figure 15. Join the ends of ribbon B and A and trim until they line up with the wrap area of ribbon B (third image in Figure 15). Then fold ribbon C towards D at the back,

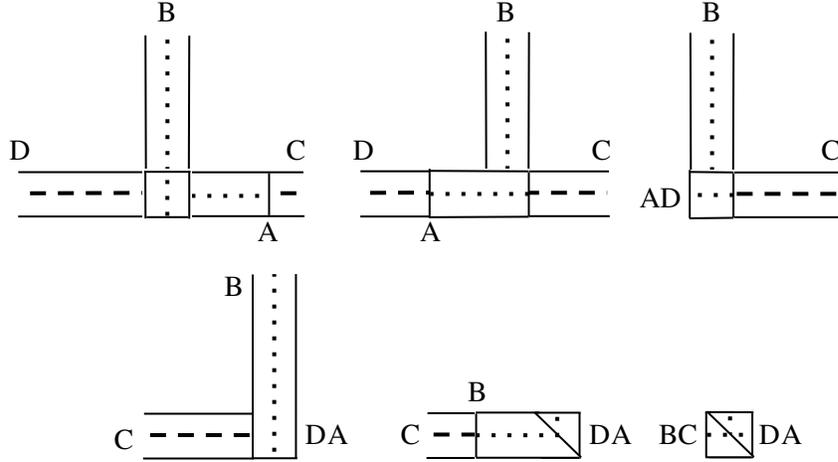


FIGURE 14. Steps of constructing $T_{2,q}$ torus knot using wrap method, where q is odd.

illustrated by the fourth image in Figure 15. Lastly, join ribbon C and D and trim until they line up with the wraps (last image of Figure 15).

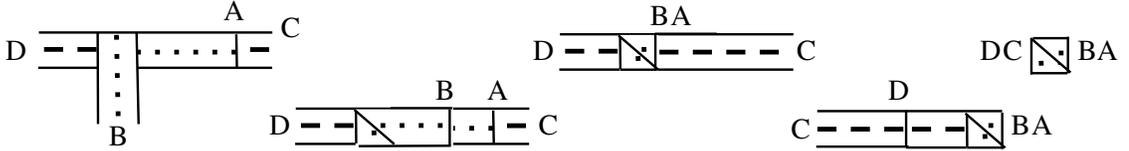


FIGURE 15. Steps of constructing $T_{2,q}$ torus knot using wrap method, where q is even.

Theorem 15. *For any $T_{2,q}$ torus link, the folded ribbonlength is $\text{Rib}([T_{2,q}]) \leq q + 3$.*

Proof.

In Construction 13 and Construction 14, we have constructed $T_{2,q}$ torus links with odd and even numbers of half-twists. We now show that for these links that we have $\text{Rib}(T_{2,q}) = q + 3$. Based on Lemma 11, we know that n half-twists can be constructed by $n + 2$ units of ribbonlength. Hence, we only need to be concerned about the ribbonlength used to join the ends to form the torus link $T_{2,q}$.

Case 1: q is odd. Following Construction 13, when folding ribbon A towards D , one unit of ribbonlength is used (one square fold). Joining and trimming AD to the wraps adds no extra ribbonlength. When we fold ribbon B towards C with $\pi/2$ fold, we preserve the one unit ribbonlength already taken into account from wrapping ribbon B , illustrated by the middle-bottom image

in Figure 14. Similarly, joining and trimming BC do not add any ribbonlength. Therefore only one more unit of ribbonlength is added while connecting the ends. Therefore, we have proved that the ribbonlength of this $T_{2,q}$ is $q + 3$.

Case 2: q is even. Following Construction 14, when we fold ribbon B towards A with $\pi/2$ fold, the one unit ribbonlength used has already been taken into account during the wrapping of ribbon B , illustrated by the second image of Figure 15. Joining and trimming BA add no extra length. One unit of ribbonlength is added as C crosses from the back to join with D (fourth image), and no more ribbonlength is added as DC shrinks to rest against the wraps of B (final image of Figure 15). Therefore only one unit of ribbonlength is added while connecting the ends. Thus, the ribbonlength of $T_{2,q}$ is $q + 3$. \square

Corollary 16. *The number of sticks of a wrap construction of a $T_{2,q}$ torus link is equal to $q + 5$.*

Proof. Based on Lemma 11, we know that n half-twists can be constructed by $n + 3$ number of sticks. Therefore, we only need to calculate the number of sticks used when joining the ends of the wrap.

Case 1: Let q be odd. The joining shown in Construction 13 shows that one stick is added as A crosses to D (second image in Figure 14), and one is added to create a $\pi/2$ fold to connect B to C (middle-bottom image in Figure 14). Therefore the knot has a number of sticks of $q + 5$.

Case 2 : Let q be even. Following Construction 14, we add one stick to create the $\pi/2$ fold to connect ribbon B to A , as shown by the second image of Figure 15. We need another stick when connecting C and D behind the wraps of B , demonstrated by the fourth image of Figure 15. Hence, 2 more sticks are added joining the ends when q is even. Therefore we have number of sticks equal to $q + 5$ as well. \square

Corollary 17. *For any torus link $T_{2,q}$, the folded ribbonlength is $\text{Rib}([T_{2,q}]) \leq \text{Cr}(T_{2,q}) + 3$*

Proof. For any torus links $T_{2,q}$, the crossing number $\text{Cr}(T_{2,q}) = q$. From Theorem 15, we know that $\text{Rib}([T_{2,q}]) \leq q + 3$. Therefore, we have proved that $\text{Rib}([T_{2,q}]) \leq \text{Cr}(T_{2,q}) + 3$ \square

Corollary 18. *For any two component link L with linking number $|\text{Lk}(L)| = n$, we have $\text{Rib}([L]) \leq 2n + 3$.*

Proof. For all two components links $L = J \cup K$, with linking number $|\text{Lk}(L)| = n$, there exists a $T_{2,2n}$ torus link with $\text{Lk}(T_{2,2n}) = n$ by Section 1.1. Based on Theorem 15, the ribbonlength for $T_{2,2n}$ satisfies $\text{Rib}(T_{2,2n}) \leq 2n + 3$. Therefore, for any link with linking number n , we see that $\text{Rib}(L) \leq 2n + 3$. \square

Following our Construction 13 and Corollary 17, we have discovered another way to build the trefoil knot where $\text{Rib}([3_1]) \leq 6$. This is the third way to construct a trefoil knot with ribbonlength 6. Denne et al. [3] have discovered two other ways to construct a trefoil knot with 6 unit of ribbonlength. Our result for $T_{2,q}$ torus knots or links is $\text{Rib}([T_{2,q}]) \leq q + 3$, while Denne et al. have $\text{Rib}([T_{2,q}]) \leq 2q$. We can observe that as q becomes larger than 3, our result are better. Previously, the general result about ribbonlength upper bound only applies to knots. In this paper, we are the first ones to generalize results to links following the previous corollaries.

4.1. Twist Knots.

Construction 19 (T_n wrap). To construct a folded ribbon twist knot T_n , we first create the number of half-twists n using the wrap method. We then connect the ends of the two parallel strands to form the clasp region, as shown in Figure 6.

Case 1: n is even. Following Construction 10 and the left-most image in Figure 16, we completed n (even) twists using the wrap method. Our goal is to join the ends of the ribbons at A and C as well as at B and D to create the clasp for the twist knots. First, fold ribbon A towards D so that the ribbon A is on top of B , shown in the second picture of Figure 16. Then, fold ribbon B upwards so that the ribbon lies over the wrap region and A , illustrated in the next picture. Next, fold ribbon C towards A and D . Shown by the rightmost picture in Figure 16, we can see that ribbon C lies on top of B , B lies on top of A , and D is at the bottom. This arrangement ensures piece of ribbon B is between the ribbon of A and C , creating the clasp region. Now we join ribbon A and C together and trim them, illustrated by the leftmost picture in Figure 17. Then, we fold ribbon D in the back to the right, as shown in the next picture. The third picture in Figure 17 shows the knot's appearance from the back as we flipped it over. Finally, fold D with a $\pi/2$ fold so it can be joined with ribbon B , finishing creating the clasp region, shown in the last picture of Figure 17. We can now trim the ends of the ribbon B and D along the wrap region to finish creating a folded ribbon twist knot with an odd number of twists.

Case 2: n is odd. Based on Construction 10, after folding n half-twists, ribbon B will be pointing upwards at the back instead of downwards at the front in the left-most graph of Figure 16. The construction of the clasp region is very similar to Case 1, but we have to fold the ribbon B between ribbon A and C at the back. Fold ribbon C towards D , then fold ribbon B pointing downwards at the back, and then fold A on top of B to connect with C . Similarly, we $\pi/2$ fold ribbon D in the front to join with B , finishing the clasp region.

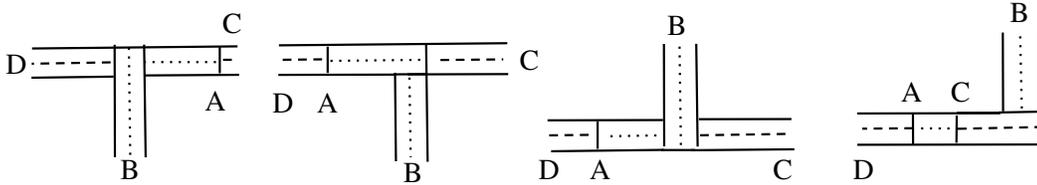


FIGURE 16. Steps of creating the clasp region for T_n .

Theorem 20. For any T_n twist knot, the folded ribbonlength is $\text{Rib}([T_n]) \leq n + 6$.

Proof. In Construction 19, we have constructed a T_n twist knot. We now show that the ribbonlength for this twist knot is $n + 6$. We know that a twist knot T_n is made up of two parts: n half-twists

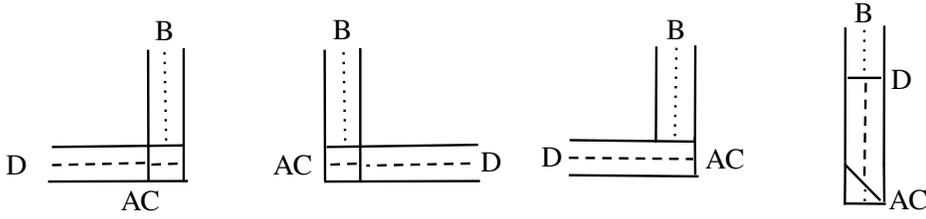


FIGURE 17. Steps of joining the clasp region for T_n .

and the clasp region; let's consider them separately. As Lemma 11 already tells us that creating n half-twists requires $n + 2$ units of ribbonlength, let's calculate how much ribbonlength we need to form the clasp region. Following Construction 19 and the leftmost picture in Figure 16, folding ribbon B in between A and C uses 3 units of ribbonlength (3 square folds). Joining A and C after the trim does not use any ribbonlength. Next, when we are trying to join ribbon B with D , the $\pi/2$ fold of D needs another 1 unit of ribbonlength. Therefore, forming the wrap region needs 4 units of ribbonlength regardless of the number of half-twists we constructed. By adding two parts together, we can now conclude that any T_n twist knot can be constructed with ribbonlength $n + 6$. \square

Corollary 21. *Number of sticks in a wrap construction of a T_n twist knot is equal to $n + 8$.*

Proof. Following Lemma 11, we know that n half-twists can be built by $n + 3$ number of sticks. Similarly, we now need to consider how many sticks are used to form the clasp region. Following Construction 19 and the left picture in Figure 16, folding ribbon B in between A and C uses 3 square folds, corresponding to 3 sticks. When connecting ribbon D to B , we have applied a $\pi/2$ fold which contains 2 sticks, as illustrated by the last two pictures in Figure 17. As 5 more sticks are used to form the clasp region, we can conclude that the number of sticks for T_n is $n + 8$. \square

Corollary 22. *For any twist knot T_n , we have $Rib([T_n]) \leq Cr([T_n]) + 4$.*

Proof. For any T_n twist knot, the crossing number $Cr(T_n)$ is $n + 2$. From Theorem 20, we know that $Rib([T_n]) \leq n + 6$. Therefore, we have proved that $Rib([T_n]) \leq Cr(T_n) + 4$. \square

Previous research [3] by Denne et al. proves that any figure eight knot has $Rib([K]) \leq 10$. Following our Construction 19, we are able to construct any figure eight knot with ribbonlength 8. We have improved the upper boundary ribbonlength of 4_1 knot from 10 to 8. The general result found in Denne et al. [3] is that for any twist knot T_n , $Rib([T_n]) \leq 2n + 6$, while in our paper, we have proved that $Rib([T_n]) \leq n + 6$. Our result is always better regardless of n .

5. PRETZEL LINKS

We are able to construct a folded ribbon 3-strand pretzel link $P_{p,q,r}$ by using a modified version of the wrap method described in Construction 10.

Construction 23 (Pretzel wrap). Begin with ribbon AB on top of and parallel to ribbon CD . Fold ribbon end B downward at angle $\pi/2$, as shown in Figure 18. Then, fold end B at angle π to result

in one positive half-twist, and repeat this move until the desired number of half-twists has been reached. Then fold end B at angle $\pi/2$ towards D . For an odd number of half-twists, end B will be under D . After an even number of half-twists, end B will be on top of D . To create negative half-twists using this method, follow the same process but begin by folding end B upward instead of downward. We will refer to this modified wrap construction as a *pretzel wrap*.

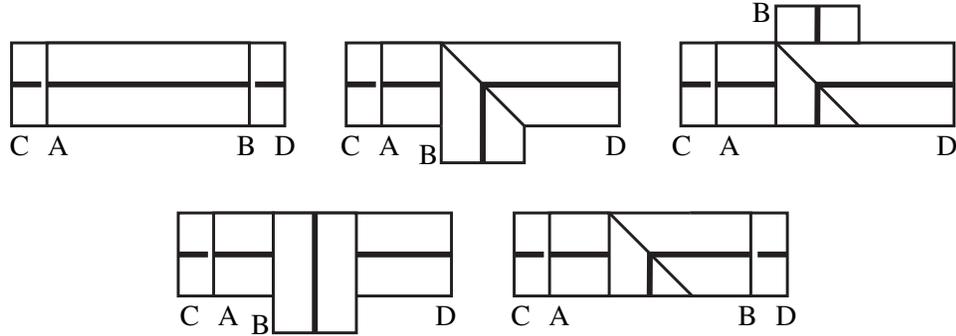


FIGURE 18. The steps to construct a pretzel wrap with two positive half-twists.

The top and bottom ends of the pretzel wrap correspond to the left and right ends of a strand of half-twists in a standard diagram, respectively. This can be seen in Figure 19, where we see two pretzel wraps and their corresponding standard diagrams. Observe that the corresponding ends for each of these constructions have been labeled accordingly. Ends A , C , E , and G are the top and left ends of the pretzel wraps and the standard strands, respectively, while ends B , D , F , and H are the bottom and right ends, respectively.

Now, as shown by both Construction 23 and Figure 19, we can see that for both pretzel wraps and standard strands of half-twists, an odd number of half-twists results in an end of the strand ending on the opposite side as which it started. Conversely, an even number half-twists results in an end of the strand ending on the same side as which it started for both constructions. As additionally shown in Figure 19, this fact does not change depending the use of positive or negative half-twists in either construction.

After constructing a pretzel wrap of one half-twist, the two ribbon strands can be easily pulled apart, which may make the construction seem trivial. However, when considering how the ends of the pretzel wrap join with the ends of other pretzel wraps, as shown in Construction 24, it will be seen that a wrap of one half-twist is different than a wrap with no twists.

Construction 24 (Pretzel link $P_{p,q,r}$). To construct a folded ribbon 3-strand pretzel link $P_{p,q,r}$, begin by constructing three pretzel wraps with p , q , and r half-twists. Then lay these wraps on top of each other so that wrap p is on top of wrap q , which is on top of wrap r . Connect the top ends of wrap r to their corresponding bottom ends of wrap q . In the same fashion, connect the top

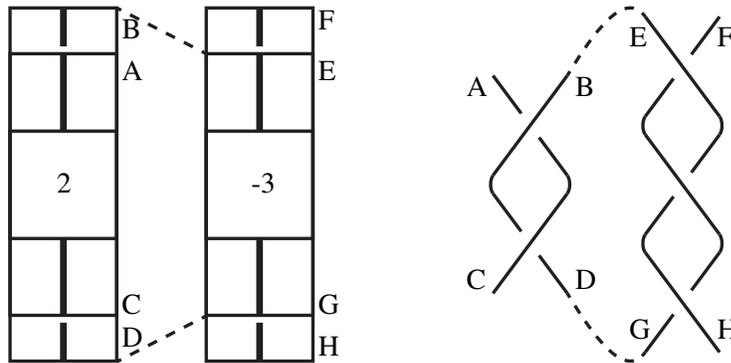


FIGURE 19. The top and bottom ends of a pretzel wrap (left) correspond to the left and right ends of a standard strand of half-twists (right).

ends of wrap q to the bottom ends of wrap p . Then connect the bottom ends of wrap r to their corresponding top ends of wrap p .

An example of this join can be seen on the left of Figure 19, where the dotted lines show how the bottom ends B and D join to the top ends E and G . The relationship between this joining method and the joining method for two standard strands of half-twists can also be seen in Figure 19, as shown by the dotted lines.

Minimize the length of these connections until the link is in the shape of a square. The relationship between this construction method and the standard diagram for a pretzel link can be seen in Figure 20. Observe that the top, middle, and bottom wraps of our construction method correspond to the left, middle, and right strands of the standard diagram, respectively. This can be seen by rotating the folded ribbon pretzel link one quarter-turn counter-clockwise.

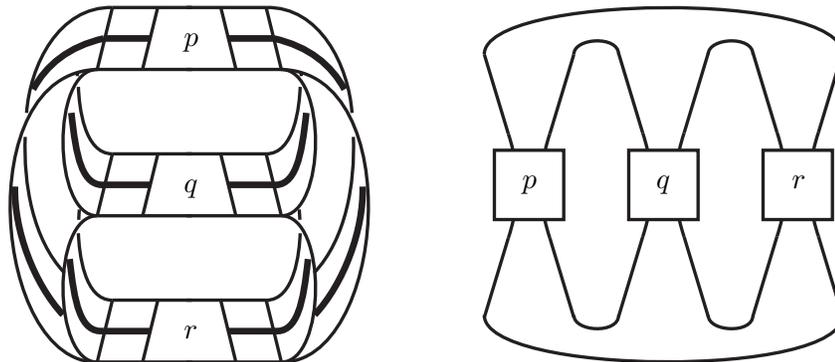


FIGURE 20. An expanded view of the construction method for a folded ribbon pretzel link with p, q, r number of half-twists (left) and its corresponding standard link diagram (right).

Theorem 25. *For any 3-strand pretzel link $P_{p,q,r}$, the folded ribbonlength is given by $\text{Rib}([P_{p,q,r}]) \leq |p| + |q| + |r| + 6$.*

Proof. Recall from Remark 3 that both a square and a pair of right isosceles triangles have ribbonlength 1. To construct one pretzel wrap, we start with the static piece of ribbon that will be wrapped around, which is a square with ribbonlength 1. With the second ribbon piece, we start with a fold at angle $\pi/2$ to give two isosceles right triangles, which brings the total ribbonlength to 2. From here we add one fold at angle π and one unit of ribbonlength for each desired half-twist. The final fold at angle $\pi/2$ is made from the last of these square units, meaning no ribbonlength is added. Hence the ribbonlength required to construct one pretzel wrap of n half-twists is $|n| + 2$. We use three of these wraps for our 3-strand pretzel link construction, and no additional ribbonlength is required to join the ends of these wraps. Therefore the required ribbonlength for this construction is $(|p| + 2) + (|q| + 2) + (|r| + 2) = |p| + |q| + |r| + 6$, and the infimal ribbonlength for any 3-strand pretzel link is less than or equal to this expression. \square

Corollary 26. *For any n -strand pretzel link L_w with (p_1, p_2, \dots, p_n) half-twists, the folded ribbonlength is given by $\text{Rib}([L_w]) \leq (\sum_{i=1}^n |p_i|) + 2n$.*

Proof. From our proof of Theorem 25 we know that one pretzel wrap of p half-twists uses ribbonlength $|p| + 2$, and that no additional ribbonlength is required to join pretzel wraps to create a pretzel link. Using this construction method, it follows that a folded pretzel link of n strands with (p_1, p_2, \dots, p_n) half-twists uses ribbonlength $(\sum_{i=1}^n |p_i|) + 2n$, and that the infimal ribbonlength for any n -strand pretzel link is less than or equal to this expression. \square

Previous work [3] found the folded ribbonlength of any 3-strand pretzel link $P_{p,q,r}$ to be given by $\text{Rib}([P_{p,q,r}]) \leq 2(|p| + |q| + |r|) + 2$. Our result of $\text{Rib}([P_{p,q,r}]) \leq |p| + |q| + |r| + 6$ improves on this previous finding for values of p , q , and r where $|p| + |q| + |r| > 3$.

Recall from Section 1.3 that any twist knot with n half-twists can be constructed as 3-strand pretzel link $P_{n,1,1}$. Thus, a twist knot made with this construction method for 3-strand pretzel links has a ribbonlength of $|n| + |1| + |1| + 6 = n + 8$. This does not improve on the ribbonlength of $\text{Rib}([T_n]) \leq n + 6$ from Theorem 20. For the figure-eight knot, which, as noted in Section 1.2, is a twist knot of two twists, this construction method yields a folded ribbonlength of 10. This is the second method of folding a figure-eight knot with ribbonlength 10 to be found, and the first method can be found in [3].

6. CRINKLE FOLD

In this section we will introduce a new construction method, the *crinkle method*. This method gives a more detailed description of the accordion method first described by [4]. The crinkle method allows us to create an upper bound of links with half-twists without regards to number of twists. We will then apply it to $T_{2,q}$ torus knots, where q is odd and T_n twist knots where n is odd. Throughout this section, we will assume the ribbon has width $w = 1$ and that the middle line of the ribbon is a part of the knot diagram.

6.1. Crinkle Folds.

Construction 27 (Crinkle folds). Take a piece of ribbon. Fold it $\pi/2$ upward so that it is perpendicular to the original piece of ribbon. Now fold it $\pi/2$ the opposite way so that it is parallel to the original piece of ribbon (see Figure 21). Repeat until you have the desired amount of folds.

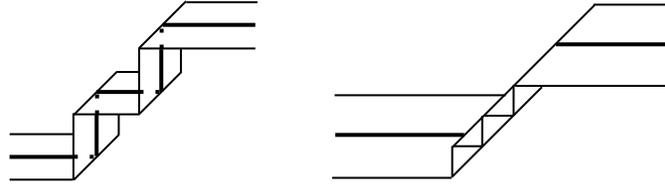


FIGURE 21. Left image shows a crinkle fold. Right image shows a tight folded crinkle fold.

Assume each fold of the ribbon is uniformly spaced with a distance d between where the the vertices of the knot turn at each fold (see left image of Figure 22). We call such a fold a *crinkle fold*. Each crinkle fold adds d to the ribbonlength.

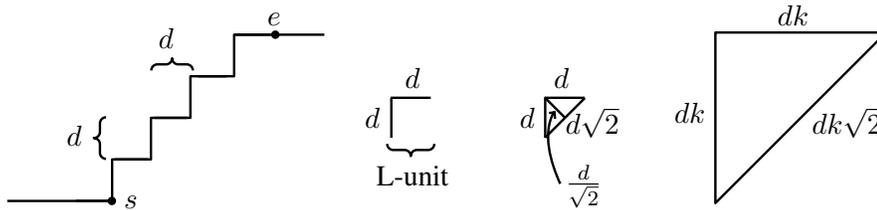


FIGURE 22. Break down of crinkle fold. Left image shows the path of the knot diagram during a crinkle fold. Second image shows one L-unit. Third image shows distances on a L-unit. Right image shows straight line and zigzag distances.

Definition 28 (L-unit). In a crinkle fold, we define an *L-unit* to be each set of two folds that return the ribbon parallel to its original position. (Second image of Figure 22). The number of L-units is denoted by k .

For each L-unit, the length of the hypotenuse of the triangle with side-length d formed by the L is equivalent to $d\sqrt{2}$ (see third image of Figure 22). The perpendicular distance from the hypotenuse to the opposite vertex of the triangle is $d/\sqrt{2}$.

The distance travelled from s to e in the direction perpendicular to the original knot diagram is the distance of each fold, d , times k , the amount of folds in that direction (right image of Figure 22). The distance from s to e in the direction parallel to the original knot diagram is also d times k . The

distance of the knot diagram from start, s , to end, e , can be seen as a combination of the distance of folds in both the parallel and perpendicular directions.

Definition 29 (Crinkle distances). Let s be the start point of a crinkle fold on a knot diagram and e be the endpoint after k L-units (see Figure 23). The distance in the plane between s and e is the *straight-line distance*, S_{se} . The distance along the knot diagram from s to e , corresponding with the ribbonlength of the ribbon, is the *zigzag distance*, Z_{se} (middle image of Figure 23).

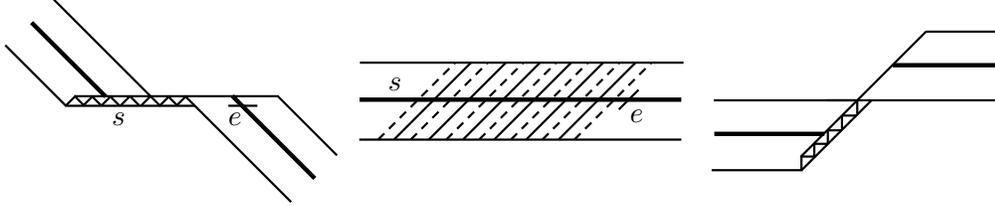


FIGURE 23. Distance of knot diagram for a crinkle fold. Left image shows straight distance, center shows zigzag distance through a crease diagram. Right image shows *escape accordion*.

The center image of Figure 23 shows a crease diagram of a crinkle fold. This is a diagram of the folds unfolded and flattened out with lines to show where the folds were. This is an additional diagram to see the zigzag distance as the length Z_{se} from s to e can be seen as the distance between them when the folds have been unfolded.

Definition 30. An *escape accordion*, seen in the right image of Figure 23, consists of a crinkle fold of straight-line distance of $S_{se} \geq \sqrt{2}$ so that the two parallel ends of the folded ribbon do not overlap.

For an escape accordion, the ends of the folded ribbon are offset enough so that the ribbon can wrap around itself without wrapping a non-crinkle fold portion of the ribbon. This means the straight-line distance must be greater than or equal to $\sqrt{2}$ as we assumed the width is 1.

Lemma 31. Take a construction of crinkle folds with a k number of L-units that start at s and end at e . The straight-line distance is $S_{se} = dk\sqrt{2}$. The zigzag distance is $Z_{se} = 2kd$. The escape accordion has a $Z_{se} = 2$ and $S_{se} = \sqrt{2}$.

Proof. The straight-line distance between s and e can be seen as the hypotenuse of the folds as seen in the right image of Figure 22. Therefore, as the hypotenuse of an L-unit is $d\sqrt{2}$ as seen in the third image of Figure 22, we see the total distance of the k L-units being $S_{se} = kd\sqrt{2}$.

The zigzag distance can be calculated as the length of folds in one direction plus the length of folds in the other direction. As the length of folds in one direction is dk (see right image of Figure 22, we see that the zigzag distance is $Z_{se} = 2dk$.

For an escape accordion, we see that that the parallel portions of the knot diagram must at least 1 unit of distance apart perpendicularly in order for them to not overlap. Therefore, the straight line

distance must be equal to $S_{se} = \sqrt{2}$. To make $S_{se} = \sqrt{2}$, we set d to a constant number. Therefore there is a certain value of k such that $dk\sqrt{2} = \sqrt{2}$, so $k = \lceil 1/d \rceil$. Then, we choose a d such that $kd = 1$. Therefore, the zigzag distance is $Z_{se} = 2$ as $Z_{se} = 2dk = 2$. \square

After the escape accordion, we can continue to fold crinkle folds. Each crinkle fold of d will continue to add $d/\sqrt{2}$ to the straight-line distance and d to the zigzag distance.

Construction 32 (Crinkle wrap). Take a construction of a crinkle fold with length long enough for an escape accordion; the two ends run parallel and they do not overlap (see left image of Figure 24). Make another $\pi/2$ fold so that the piece of ribbon wraps under and up. Here the ends of the ribbon are now perpendicular. This is one half-wrap (middle image of Figure 24). Fold the same end of the ribbon back down so the two ends are parallel for two half-wraps (right image of Figure 24). Repeat for n half-wraps.

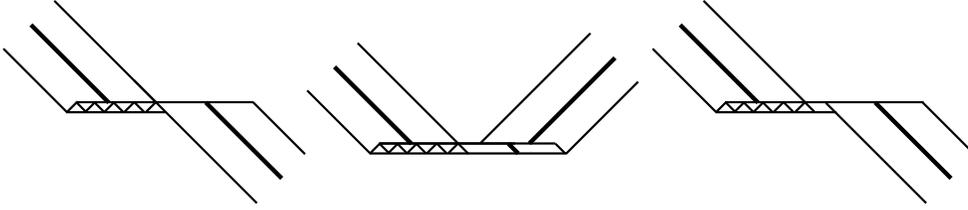


FIGURE 24. Construction of crinkle wrap. Left shows escape accordion. Center shows first half-wrap. Right shows two half-wraps.

Lemma 33. For an odd amount of wraps (detailed in Construction 32), the straight line distance is $S_{se} = \sqrt{2} + \frac{n-1}{2}d\sqrt{2}$ and the zigzag distance is $Z_{se} = 2 + (n-1)d$.

Proof. Assume we have an odd number of half-wraps that have perpendicular ends (detailed in Construction 32). They go from s , the beginning of the crinkle folds, to e_i , the end of the wraps along the bottom edge of the wraps, where i is the number of wraps (left image of Figure 25). Therefore e_0 corresponds to only the escape accordion, e_1 to escape accordion plus one wrap, e_2 to two wraps, etc..

Looking at the middle image of Figure 25, we see that for 1 half-wrap, the straight-line distance from s to e_1 is the same as the escape accordion at $S_{ae_0} = \sqrt{2}$, a distance that was proven in Lemma 31. This is because the distances of d place e_0 at the border of the wraps, so that when we create the first wrap, no extra length is added. Therefore $S_{ae_1} = \sqrt{2}$. Then, looking at the right image of Figure 25, we see that for 3 half-wraps, e_i only moves the distance of one $d\sqrt{2}$ from e_0 to e_3 . For each additional 2 half-twists, which is equivalent to one L-unit, the overall straight-line distance increases by $d\sqrt{2}$. Therefore, for an odd amount of wraps, the straight-line distance is $S_{se} = \sqrt{2} + \frac{n-1}{2}d\sqrt{2}$.

For the zigzag distance, we see a similar pattern. With the escape accordion, the zigzag distance is $Z_{se_0} = 2$ from s to e_0 . For the first wrap, no zigzag distance was added as e_0 is already in line

with the border of the wraps. Then, each wrap adds one distance d to Z_{se_i} . Therefore, the zigzag distance is $Z_{se} = 2 + (n - 1)d$. \square

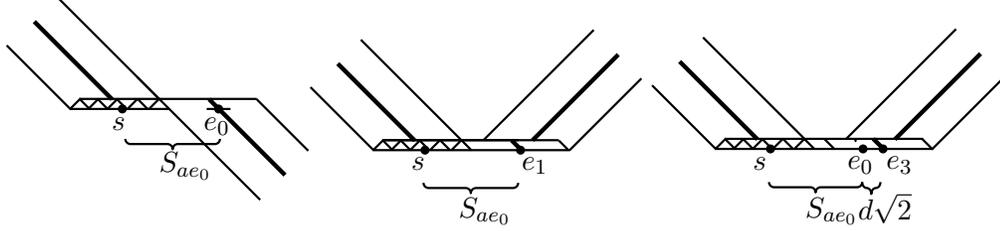


FIGURE 25. Figure shows odd half-wraps and their distances. Left shows escape accordion. Middle shows 1 half-wrap. Right shows 3 half-wraps.

Corollary 34. For an odd number of crinkle folds after an escape accordion, $S_{se} = \sqrt{2} + \frac{n-1}{2}d\sqrt{2}$ and $Z_{se} = 2 + (n - 1)d$.

Proof. We use the same notation for s and e_i as in Lemma 33. For an odd number of crinkle folds, we see that as we look at the straight-line distance at the bottom of the folds, e_0 and e_1 are the same, just as it is for an odd number of wraps. Therefore, we see that for crinkle folds, the distances are also $S_{se} = \sqrt{2} + \frac{n-1}{2}d\sqrt{2}$ and $Z_e = 2 + (n - 1)d$. \square

6.2. Crinkle $T_{2,q}$ Odd Torus Knots.

Construction 35 ($T_{2,q}$ odd crinkle). To form a $T_{2,q}$ torus knot, where q is odd, we will have to create q half-twists and join opposite corresponding ends together. We see this in the left image of Figure 26 where C and B must join together, and A and D must connect. To create q half-twists with crinkle folds, we have two constructions of the crinkle fold, CD and AB , where AB begins as an escape accordion, and CD has an escape accordion and more length of crinkles so its straight-line distance is longer than the length of the folds of AB from outer edge to outer edge. Then we wrap have AB wrap around both of them while CD continues to crinkle.

To start, take two constructions of a crinkle fold: one, AB , an escape accordion, and the other, CD , with a straight-line distance from s to e longer than the length of the folds of AB from outer edge to outer edge. We make the ends of AB parallel to each other and have CD have an odd number of folds so that the ends are perpendicular to each other (see right image of Figure 26). Then we place the piece of ribbon AB on top of CD such that A lies on top of C and the folds of AB are directly on top of the folds of CD (see left image of Figure 27). Now, we start wrapping end B around both itself and the folds of CD , starting with wrapping underneath and upward, and wrap any odd number of half-wraps (right image of Figure 27). As we wrap end B , we continue to create crinkles in end D so that end B and end D never overlap. Each of these half-wraps of end B around CD correspond to one half-twist between AB and CD .

To create the connections of C to B and A to D , we start by folding the piece of ribbon C for $\pi/4$ backward so that it is parallel with the set of wraps and the edge of the piece of ribbon C is

directly below the lower edge of the folds (see left image of Figure 28). Next we fold end B for $\pi/4$ backward and join end B and end C together. Now, fold end A for $\pi/4$ backward directly underneath and in line with CB . Then fold end D for $\pi/4$ backward so that D is underneath and in line with A . Now we join end A and end D together (right image of Figure 28).

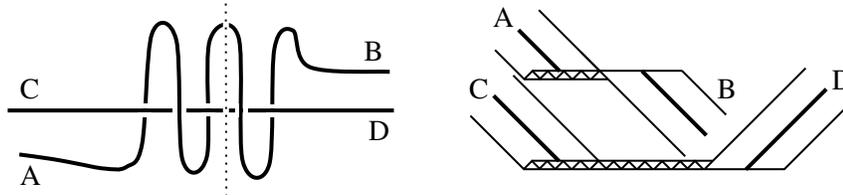


FIGURE 26. Left image is a link diagram with an odd number of half-twists. Right image shows two crinkle fold constructions, AB and CD .

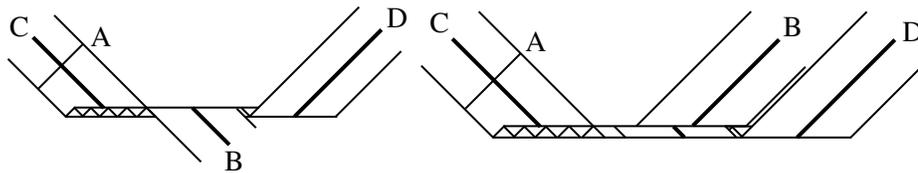


FIGURE 27. Left image shows AB layered on top of CD . Right image shows wrapping of AB around CD .

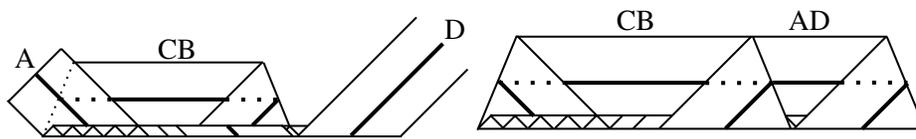


FIGURE 28. Joining of two wrapped crinkle folds. Left image shows joining of CB . Right image shows joining of AD .

Remark 36. When constructing the wraps of B around CD , the piece of ribbon CD must have a longer straight-line fold distance to allow it to escape from the wraps of B (right image of Figure 26). As the straight-line width of B while folding is $\sqrt{2}$ due to it being 1 unit of width at a $\pi/4$ diagonal, we see that for 1 wrap, the straight-line distance of AB from the start of the folds at A , s_A to the end of the folds at B , e_B is $S_{s_A e_B} = \sqrt{2}$, while the straight-line distance of CD goes from the start of the folds at C , s_C to the end of the folds at D , e_D , with $S_{s_C e_B} + S_{e_B e_D} = 2\sqrt{2}$. The separation into $S_{s_C e_B}$ and $S_{e_B e_D}$ show how the folds of CD must go a $\sqrt{2}$ past the folds of AB . Therefore, this second escape accordion for CD has a straight-line distance of $S_{e_B e_D} = \sqrt{2}$. As $S_{e_B e_D} = \sqrt{2}$ the zigzag distance of this second escape for CD is $Z_{e_B e_D} = 2$.

Proposition 37. To find the ribbonlength of pieces CD joined to AB , we observe Figure 29. We find the length of the knot diagram by following the labelled points that show a break down of all the distances of a joined crinkle fold. All of the distances are measured along the knot diagram of the folded ribbon. The distance \overline{bc} is the straight-line distance of the fold, S_{se} . We can see that since triangle abg has an angle of $\pi/4$ and that the distance $\overline{gb} = 1/2$, the distance $\overline{ag} = 1/2$ (see Figure 30). We can also see that $\overline{ab} = 1/\sqrt{2}$.

Corollary 38. For a joined crinkle fold with labels detailed in Figure 29, the distance $\overline{ge} = \overline{fh} = 1/\sqrt{2}$.

Proof. For $\overline{ge} = \overline{fh}$, we can think of the fact that the original escape accordion has a straight distance of $\sqrt{2}$. When it was just the escape accordion, j and k touched. Therefore the distance of $\overline{ge} + \overline{fh} = \sqrt{2}$, thus we can see $\overline{ge} = \overline{fh} = 1/\sqrt{2}$. \square

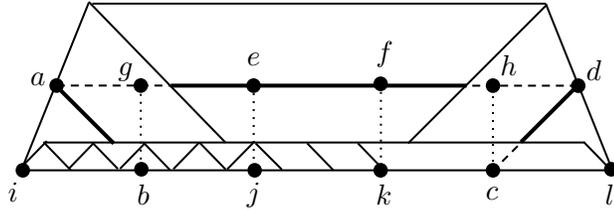


FIGURE 29. Break-down of the distances of a joined crinkle fold.

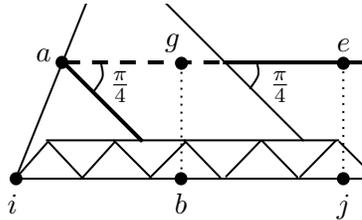


FIGURE 30. Close up on left angle of joined crinkle fold.

Theorem 39. *Given $T_{2,q}$ torus knot where q is odd, then $\text{Rib}([T_{2,q}]) \leq 16$.*

Proof. From Construction 35, we have a $T_{2,q}$ torus knot where q is odd. Now we will add up all the separate distances we see in Figure 29 and detailed in Proposition 37 and Corollary 38 applied to the connection detailed in Construction 35. As the joined fold is completely symmetrical, all distances on the left side are equivalent to the distances on the right side. For all distances except \overline{ef} and \overline{bc} , the distances on all the pieces of ribbon A, B, C , and D are the same. To notate the location of points on the different pieces of ribbon, we will use subscripts. For example, a_A and a_C are points on top of each other, but a_A is located on A and a_C is located on C . Therefore, we have, for q odd, $\text{Rib}(T_{2,q}) = 4\overline{ab} + 4\overline{ag} + 4\overline{ge} + \overline{e_C f_B} + \overline{e_A f_D} + Z_{b_A c_B} + Z_{b_C c_D}$. The distances of $\overline{e_C f_B}$ across CB and $\overline{e_A f_D}$ are equal to straight-line distance created with the amount of wraps or folds, disregarding the distance from the escape accordian (which is already accounted for). Therefore, we see that $\overline{e_C f_B} = S_{e_C f_B} = \frac{n-1}{2}d\sqrt{2}$ from Lemma 33. For \overline{ef} along AD , we see that we also have the second escape accordian for CD to account for in the straight-line distance. Therefore the distance $\overline{e_A f_D} = S_{e_A f_D} = \sqrt{2} + \frac{n-1}{2}d\sqrt{2}$ from Corollary 34 and Remark 36. The zigzag distance across AB is $Z_{b_A c_B} = 2 + (n-1)d$, as seen in Lemma 33. The zigzag distance across CD is $Z_{b_C c_D} = 2 + (n-1)d + 2$, as seen in Corollary 34 and Remark 36.

Plugging in these values and the ones from Proposition 37 and Corollary 38 we see that

$$\begin{aligned} \text{Rib}(T_{2,q}) &\leq 4\overline{ab} + 4\overline{ag} + 4\overline{ge} + \overline{e_C f_B} + \overline{e_A f_D} + Z_{b_A c_B} + Z_{b_C c_D} \\ &\leq 4\frac{1}{\sqrt{2}} + \frac{4}{2} + 4\frac{1}{\sqrt{2}} + \frac{n-1}{2}d\sqrt{2} + \sqrt{2} + \frac{n-1}{2}d\sqrt{2} + 2 + (n-1)d + 4 + (n-1)d \\ &\leq 8 + 5\sqrt{2} + (n+1)d\sqrt{2} + 2(n-1)d \\ &\leq 15.0711 + d[(n+1)\sqrt{2} + 2(n-1)]. \end{aligned}$$

The second term of this inequality is essentially d times a constant, X . This X is dependent on n , the number of wraps. For any $\varepsilon < 0$, we can make $d \cdot X < \varepsilon$ by choosing a d such that $d < \frac{\varepsilon}{X}$. Thus we can choose an ε such that $\text{Rib}(T_{2,q}) = 16$, for example. We could also make this lower than say 15.1 by choosing a smaller ε .

From this construction we have proved the $\text{Rib}([T_{2,q}]) \leq 16$, for q odd. \square

There may be other ways to make $\text{Rib}([T_{2,q}]) \leq 15.1$ smaller.

Theorem 40. *Given the equation $c_1 \cdot (\text{Cr}(K))^\alpha \leq \text{Rib}([K]) \leq c_2 \cdot (\text{Cr}(K))^\beta$, the exponent $\alpha = 0$.*

Proof. Using our results from Theorem 39, we see that the $\text{Rib}([T_{2,q}]) \leq 16$ for all odd q 's. As this is independent of the crossing number, the exponent of the crossing number of the lower bound of the ribbonlength, α , must be equal to 0. \square

6.3. Crinkle T_n Odd Twist Knots. Observe the left image of Figure 31. This is a twist knot T_n of an odd n . This is composed of a set of half-twists on the bottom of the image, and a clasp region on the top of the image. We can use Construction 35 to create n half-twists then create a clasp while joining the ends. Looking at the right image of Figure 31, we see a depiction of a twist knot. This depiction represents how a clasp must be constructed while using crinkle folds. Due to the two

pieces of ribbon with the crinkle folds being directly on top of each other, a twist is created in the knot diagram to represent the way the ribbons interact without having the diagram overlap itself.

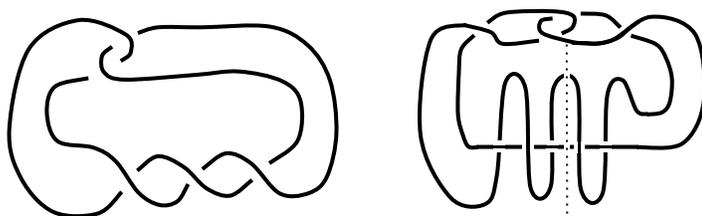


FIGURE 31. Left image shows T_3 twist knot. Right image shows T_n twist knot of odd n for crinkle joining.

Construction 41. To start, take two constructions of a crinkle fold the same as in Construction 35. We continue to follow Construction 35 until we have created the desired amount of half-wraps of B ; stopping before joining the ends (see left image of Figure 32).

Fold the piece of ribbon A for a $\pi/4$ fold such that it lays over B and D and the lower edge of A is directly above the lower edge of the folds (right image of Figure 32). Now fold end B directly downward over top A (left image of Figure 33). Next, fold end C directly over A , crossing over B (right image of Figure 33). Join the ends of C and A and shrink the join so it lays against the edge of B . Now, fold end D over the clasp region so it lays directly on C (left image of Figure 34). Lastly fold end D downward for $\pi/2$ and join end D and end B together (right image of Figure 34).

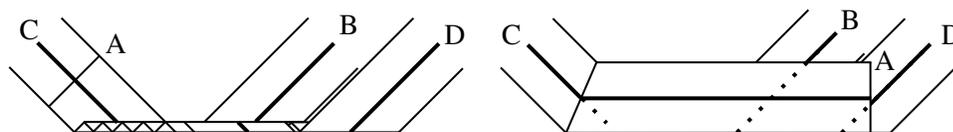


FIGURE 32. Start of crinkle twist knot construction. Left image shows set up after following Construction 35. Right image shows folding A overtop the folds.

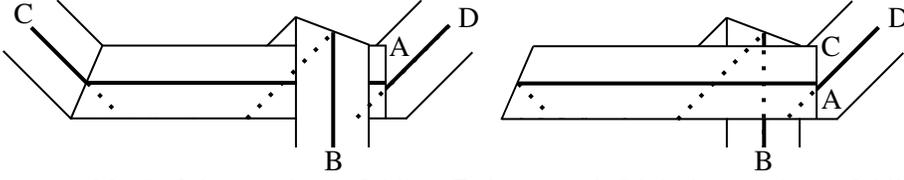


FIGURE 33. Left image shows folding B downward. Right image shows folding C overtop A .

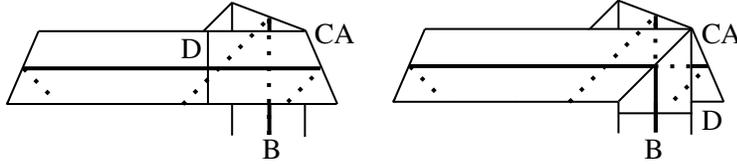


FIGURE 34. Left image shows folding D overtop the clasp. Right image shows folding D downward.

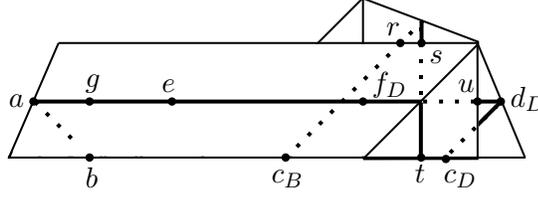
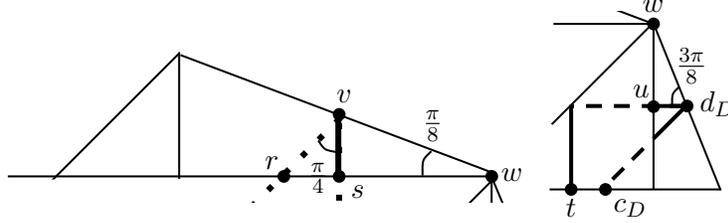
Now we will start to compute the ribbonlength of Construction 41. To start, we will add the clasp region to the points and distances detailed in Figure 29. To do so, we will continue with the subscript notation with the already assigned points, and add points r, s, t , and u to detail the distances of the clasp region, as we can see in Figure 35.

Lemma 42. *When looking at the diagram of a completed T_n twist knot where n is odd, (see Figure 35), the distance along the knot diagram from r to s is $\overline{rs} = \frac{1}{2}$ and the distance from u to d_D is $\overline{ud_D} = \frac{1}{2\sqrt{3+2\sqrt{2}}}$.*

Proof. To find the distance along the knot diagram for r to s and u to d_D , we start by observing Figure 36, which zooms in on these distances and shows the values of the surrounding angles. We add the extra points v and w to help describe the distances. From Construction 41, we know that angle $\angle rvs = \pi/4$. Due to the nature of a fold being a mirror, we know that angle $\angle wvs = \frac{\pi - \pi/4}{2} = 3\pi/8$. Therefore, angle $\angle vws = \pi/8$. As in Construction 41 the angle $\angle ud_D u$ is the same $\pi/4$ fold as $\angle rvs$, we see that $\angle wd_D u = 3\pi/8$, just as $\angle wvs$.

As the point s lies on the knot diagram, and we assume the ribbon to have width 1, the distance from s to w is $1/2$. Therefore, the distance from s to v is $\tan(\pi/8) = \overline{sv}/(1/2)$. Thus $\overline{sv} = \frac{\sqrt{2}-1}{2}$. The distance from v to r can then be seen as $\cos(\pi/4) = \overline{sv}/\overline{vr}$. Thus $\overline{vr} = \frac{(\sqrt{2}-1)/2}{1/\sqrt{2}} = \frac{2-\sqrt{2}}{2}$. Therefore, we can see that the distance along the knot diagram from r to s , or $\overline{rs} = 1/2$.

For the distance from u to d_D , we see that the distance from u to w is $1/2$. Therefore we see that $\tan(3\pi/8) = \frac{1/2}{\overline{ud_D}}$. Thus $\overline{ud_D} = \frac{1}{2 \tan(3\pi/8)} = \frac{1}{2\sqrt{3+2\sqrt{2}}}$. \square

FIGURE 35. Crinkle T_n twist knot with break-down of distances.FIGURE 36. Left image is a zoom in on top right corner to show the distance from r to s . Right image is a zoom in on the right side to show the distance from u to d_D .

Theorem 43. *Given T_n twist knot where n is odd, then $\text{Rib}([T_n]) \leq 20$.*

Proof. From Construction 41, we have a T_n twist knot where n is odd. Now we will add up the distances found in Figure 35 with respect to Figure 29, Proposition 37, Corollary 38, and Lemma 42. The construction of a T_n twist knot, with n odd, is not completely symmetrical, so we will have to detail the distances for each piece and end of ribbon. For the folds and wraps of AB , we see a ribbonlength of $Z_{b_Ac_B}$, and for the folds of CD , we see a ribbonlength of $Z_{b_Cc_d}$.

For end A , it travels from the beginning of the folds, b to a , then to g , e , f_D , and ending at u where it connects to C . End C follows the exact same path from b to a , g , e , f_D , then finally to u where it connects to A .

End B goes from the end of the wraps of B , c_B to r , then to s and down to t , where it connects to D . End D goes from the end of its crinkle folds, c_D , to d_D , then to u , where it gets folded down to t to connect to B .

Therefore, for the ribbonlength, we have the distances:

$$\text{Rib}([T_n]) = Z_{b_Ac_B} + Z_{b_Cc_d} + 2[\overline{ab} + \overline{ag} + \overline{ge} + \overline{ef_D} + \overline{f_Du}] + \overline{c_Br} + \overline{rs} + \overline{st} + \overline{c_Dd_D} + \overline{d_Du} + \overline{ut}.$$

We know the distances of $Z_{b_Ac_B}$ and $Z_{b_Cc_d}$ from Lemma 33 and Corollary 34; \overline{ab} , \overline{ag} , \overline{ge} , and $\overline{c_Dd_D}$ from Proposition 37 and Corollary 38; and \overline{rs} and $\overline{ud_D}$ from Lemma 42. We can see that $\overline{ef_D}$ is the straight-line distance of the folds of CD , not including the first escape accordion, which is $S_{e_Cf_D} = \sqrt{2} + \frac{n-1}{2}d\sqrt{2}$. From f_D to u , is the width of end A , which we have assumed is 1. This is the same for the distance from s to t , which is also the width of the ribbon, 1. The distance from u to t is also 1 based on Remark 3. The distance from c_B to r is $\sqrt{2}$, as end B is at a $\pi/4$ angle from A and C .

Therefore, when we plug all values in, we get the ribbonlength is

$$\begin{aligned} \text{Rib}([T_n]) &\leq Z_{b_Ac_B} + Z_{b_Cc_D} + 2[\overline{ab} + \overline{ag} + \overline{ge} + \overline{ef_D} + \overline{f_Du}] \\ &\quad + \overline{c_Br} + \overline{rs} + \overline{st} + \overline{c_Dd_D} + \overline{d_Du} + \overline{ut} \\ \text{Rib}([T_n]) &\leq 2 + (n-1)d + 4 + (n-1)d + 2\left[\frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{\sqrt{2}} + \sqrt{2} + \frac{n-1}{2}d\sqrt{2} + 1\right] \\ &\quad + \sqrt{2} + \frac{1}{2} + 1 + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3+2\sqrt{2}}} + 1 \\ \text{Rib}([T_n]) &\leq 11 + \frac{1}{2} + \frac{11}{\sqrt{2}} + \frac{1}{2\sqrt{3+2\sqrt{2}}} + d[(n-1)(2+\sqrt{2})] \\ \text{Rib}([T_n]) &\leq 19.4853 + d[(n-1)(2+\sqrt{2})] \end{aligned}$$

As the second term is d times a constant, X , we can make $d \cdot X < \varepsilon$, for any $\varepsilon < 0$, by choosing a d such that $d < \frac{\varepsilon}{X}$. Thus we can choose an ε such that $\text{Rib}(T_n) = 20$, for example.

From this construction, we have proved the $\text{Rib}([T_n]) \leq 20$, for n odd. \square

We could also make this lower than say 19.5.

Through this ribbonlength result, we see that Construction 41 gives us a second knot family to prove Theorem 40.

7. ACKNOWLEDGMENTS

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