ALTERNATING QUADRISECANTS OF KNOTS

BY

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Abstract

A knot is a simple closed curve in $\mathbb{R}^3$. A secant line is a straight line which intersects the knot in at least two distinct places. Trisecant, quadrisecant and quintisecant lines are straight lines which intersect a knot in at least three, four and five distinct places, respectively. It is clear that any closed curve has secants. A little thought will reveal that nontrivial knots must have trisecants, but they do not necessarily have quintisecants. The relationship between knots and quadrisecants is not so immediately clear. In 1933, E. Pannwitz proved that nontrivial generic polygonal knots have at least one quadrisecant. In 1994, G. Kuperberg showed that all (nontrivial tame) knots have at least one quadrisecant.

Quadrisecants come in three basic types. These are distinguished by comparing the orders of the four points along the knot and along the quadrisecant line. These three types are labeled simple, flipped and alternating. It turns out that alternating quadrisecants capture the knottedness of a knot.

The Main Theorem shows that every nontrivial tame knot in $\mathbb{R}^3$ has an alternating quadrisecant. This result refines the previous work about quadrisecants and gives greater geometric insight into knots. The Main Theorem provides new proofs to two previously known theorems about the total curvature and second hull of knotted curves. Moreover, essential alternating quadrisecants may be used to dramatically improve the known lower bounds on the ropelength of thick knots.

To Michael

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Introduction

1.1 History and motivation

A knot is a simple closed curve in $\mathbb{R}^3$. A secant line is a straight line which intersects the knot in at least two distinct places. Trisecant, quadrisecant and quintisecant lines are straight lines which intersect a knot in at least three, four and five distinct places, respectively. It is clear that any closed curve has secants. A little thought will reveal that nontrivial knots must have trisecants (see Lemma 2.1.5), but they do not necessarily have quintisecants (see Proposition 2.2.8). The relationship between knots and quadrisecants is not so immediately clear.

In 1933, E. Pannwitz [Pann] first showed that generic polygonal knots in $\mathbb{R}^3$ have at least $2u^2$ quadrisecants, where u is the unknotting number[1] of a knot. In 1980, H.R. Morton and D.M.Q. Mond [MM] reworked this result and conjectured that a generic knot with crossing number n has at least $\binom{n}{2}$ quadrisecants. It was not until 1994 that G. Kuperberg [Kup] managed to extend the result and show that every nontrivial tame knot in $\mathbb{R}^3$ has a quadrisecant.

Quadrisecants come in three basic types. These are distinguished by comparing the orders of the four points along the knot and along the quadrisecant line. There are three relative

[1]The unknotting number $u(K)$ of a knot K is defined to be the minimum number of crossing changes required to change K to the unknot, where the minimum is taken over all possible sets of crossing changes in all possible diagrams of K.

orderings of an oriented quadrisecant line and an unoriented knot. Given quadrisecant *abcd* with that ordering along the quadrisecant line, then the possible dihedral orderings along the knot are *abcd*, *abdc* and *acbd*. The Main Theorem (see Theorem 4.1.3) shows that every nontrivial tame knot in $\mathbb{R}^3$ has at least one quadrisecant with knot ordering *acbd*. This type of quadrisecant will be called an alternating quadrisecant. This result refines the previous work about quadrisecants, giving greater geometric insight into knots and also providing several interesting applications. Recently, R. Budney *et al.* [BCSS] have shown that the finite type 2 Vassiliev invariant can be computed by counting alternating quadrisecants with appropriate multiplicity. While this result implies the existence of alternating quadrisecants for many knots, our Main Theorem shows existence for *all* (nontrivial tame) knots. On the other hand, our results provide no way to count alternating quadrisecants.

One application of the existence of alternating quadrisecants for knotted curves has been to better understand the curvature of knotted curves. For smooth closed curves, the total curvature is the total angle through which the unit tangent vector turns (or the length of the tangent indicatrix). Equivalently, for polygonal curves the total curvature is the sum of the exterior angles. Around 1949 I. Fáry [Far] and J.W. Milnor [Mil] proved independently that the total curvature of a nontrivial tame knot in $\mathbb{R}^3$ is greater than or equal to 4π. This result has recently (1998) been extended to knotted curves in Hadamard[2] manifolds by C. Schmitz [Schm] and S.B. Alexander and R.L. Bishop [AB]. To prove the inequality, both of these papers used the fact that the total curvature of an inscribed polygon is less than or equal to the total curvature of the curve it is inscribed in. The total curvature of an alternating quadrisecant is 4π. Thus if a knotted curve has an alternating quadrisecant, the total curvature will be greater than or equal to 4π, giving the Fáry-Milnor theorem another proof. In fact C. Schmitz nearly shows that an alternating quadrisecant exists, but in his proof, some quadrisecants may degenerate to trisecants. S.B. Alexander and R.L. Bishop

[2] A Hadamard manifold is a complete simply-connected Riemannian manifold with non-positive sectional curvature.

show that a doubly covered bi-gon is inscribed in a polygonal curve inscribed in the knotted curve.

Another application is to the second hull of a knotted curve in $\mathbb{R}^3$. Intuitively, the second hull is the part of space that the knotted curve winds around twice. In 2000, J. Cantarella *et al* [CKKS] proved that the second hull of a knotted curve in $\mathbb{R}^3$ is non-empty. This paper is where the existence of alternating quadrisecants for knotted curves in $\mathbb{R}^3$ was first conjectured. The existence of alternating quadrisecants provides another way to show that the second hull of a knotted curve is nonempty.

The second hull paper [CKKS] resulted from work looking at the ropelength of a knot [CKS]. Given a rope of radius 1, how much length of rope is needed to tie a knot? The ropelength of a knot is the quotient of its length and its thickness. The thickness is the radius of the largest embedded normal tube around the knot. The numerical value of the minimum ropelength needed to tie any nontrivial knot is not currently known. In joint work with J.M. Sullivan and Y. Diao [DDS], essential alternating quadrisecants are used to dramatically improve the known lower bounds of ropelength from 24 to 31.32. Numerical experiments by P. Pieranski [Pie] and J.M. Sullivan [Sul] have found a trefoil knot with ropelength less than 32.7, so the new bounds are quite sharp.

The proof of the Main Theorem will extend ideas taken from all of the previous papers, but in particular from G. Kuperberg [Kup], E. Pannwitz [Pann] and C. Schmitz [Schm]. At its core, the proof will assume that alternating quadrisecants do not exist, then it will use this to create a contradiction to knottedness. A quadrisecant includes a number of trisecants, thus a large part of the proof will be dedicated to a detailed understanding the structure of the set of trisecants of a knot, both in K^3 and when projected to the set of secants $S = K^2 \smallsetminus \Delta$.

Section 1.2 reviews the basic knot theory assumed and used in the rest of the work. Section 2.1 introduces n-secants and explores the relationship between quadrisecants and trisecants. Lemma 2.1.5 shows that trisecants exist for knotted curves and Lemma 2.1.6

shows that alternating quadrisecants exist when sets of trisecants of same and different orders share common points. A detailed understanding of the structure of the set of trisecants is required. In particular, this set has very nice properties when the knot is a generic polygonal knot. In Section 2.2, the definition of a generic polygonal knot is given and in Proposition 2.2.8 we prove that the set of all generic n-gons is open and dense in $\mathbb{R}^{3n}$. In Section 2.3, the structure of the set of trisecants of generic polygonal knots is examined in detail, both in K^3 and $K^2 \setminus \Delta$. In $K^2 \setminus \Delta$, it is shown to be the image of a piecewise smooth immersion of a 1-manifold, which intersects itself in double points. In the end we wish to prove the existence of alternating quadrisecants for all (nontrivial tame) knots. Any tame knot is the limit of a sequence of generic polygonal knots. Thus we first prove the existence of an essential alternating quadrisecant for nontrivial generic polygonal knots. The notion of essential is required so that quadrisecants do not degenerate to trisecants in the limit. Section 3.1 defines the notion of essential for secants, trisecants and quadrisecants and describes the structure of the set of essential trisecants. Section 3.2 gives the proof that any nontrivial generic polygonal knotted curve has an essential alternating quadrisecant. In Section 4.1, the Main Theorem is proved: Every nontrivial tame knot in $\mathbb{R}^3$ has an alternating quadrisecant. This result is strengthened in Theorem 4.1.10 which shows that *essential* alternating quadrisecants exist for nontrivial knots of finite total curvature. The Main Theorem provides new proofs to two theorems which are important results in their own right. Section 4.2 gives these new proofs. Finally in Chapter 5 we see how essential alternating quadrisecants may be used to dramatically improve the known lower bounds of ropelength of thick knots.

1.2 Preliminaries

We will begin with some basic definitions:

Definition 1.2.1. A **knot** K is a homeomorphic image of S^1 in $\mathbb{R}^3$ (or S^3), modulo

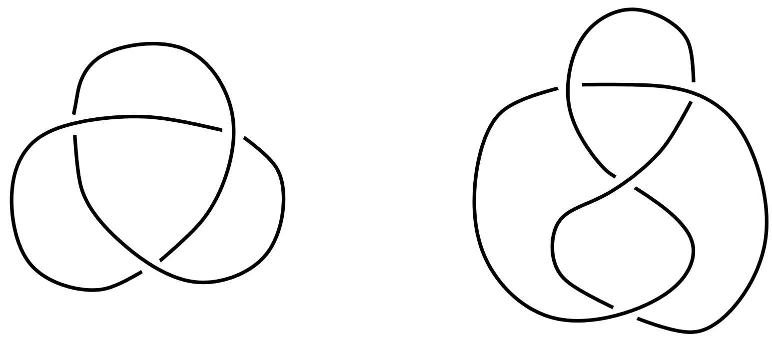

Figure 1.1: The trefoil knot (left) and the figure eight knot (right).

reparametrizations.

Other texts define a knot as a a simple closed curve in $\mathbb{R}^3$ (simple means that the curve does not intersect itself). Frequently the image of S^1, $h(S^1)$, is not distinguished by notation from the knot $K = h(S^1)$ as a subset of $\mathbb{R}^3$. Some examples of knots can be found in Figure 1.1. Knots (and links) are either defined as curves in $\mathbb{R}^3$ or S^3. This difference doesn't matter and results in this section will be stated in both settings.

Definition 1.2.2. A **link** L of m components is a subset of $\mathbb{R}^3$ of m disjoint, simple closed curves in $\mathbb{R}^3$. (When $m = 1$ we have a knot.)

One of the central questions of knot theory is to decide whether two knots are the same or different. Of course, one first must decide what is meant by "same"! Intuitively, when given a piece of string, one can move strands in space and still have the same kind of knot. It is only when one "unties" the string that the knot changes. This intuition forms the basis of our first definition of equivalence of knots.

Definition 1.2.3. A homotopy $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ is an **ambient isotopy** between K_1 and K_2 if

1) h_0 is the identity,

2) each h_t is a homeomorphism and

3) $h_1 = h$, where $h : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism such that $h(K_1) = K_2$.

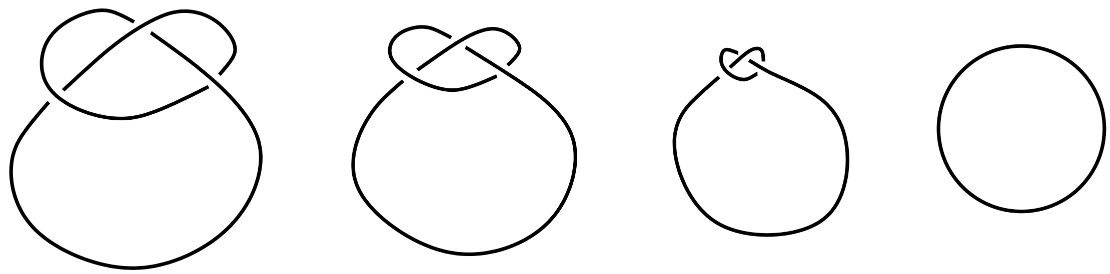

Figure 1.2: The trefoil knot is shown to be isotopic to the unknot by "pulling the knot tight" to one point on the unknot. However, the trefoil knot is not ambient isotopic to the unknot.

Definition 1.2.4. Two knots K_1 and K_2 are **equivalent** if they are ambient isotopic. The equivalence class of a knot is called its **knot type**.

Note that that the restriction of the homeomorphism $h_1 : \mathbb{R}^3 \setminus K_1 \to \mathbb{R}^3 \setminus K_2$ is also a homeomorphism if K_1 and K_2 are ambient isotopic. Also note that the definition of ambient isotopy may be extended so that:

Definition 1.2.5. Two links are **equivalent** if they are ambient isotopic.

There are several other ways that two knots may be considered equivalent. It is worthwhile discussing these definitions as it will help clarify our understanding of what we mean by a knot. Other texts ([Rolf], [GP] and [BZ]) also give excellent introductions to this topic.

A beginner in knot theory may wonder about the importance of the space $\mathbb{R}^3$ (or S^3) to the isotopy. They might think the following definition of knot equivalence is sufficient.

Definition 1.2.6. Two knots K_1 and K_2 are equivalent if there is an isotopy $H : S^1 \times I \to \mathbb{R}^3$ such that $H(\cdot, 0) = K_1$ and $H(\cdot, 1) = K_2$.

Two knots which are ambient isotopic are clearly equivalent under this definition. This definition also gives an equivalence relation on knots. However, under this definition, any two knots are isotopic to the trivial knot, hence to each other. This can be understood by the simple motion of pulling a knot "tight" continuously to a point. At each stage there will be the desired homeomorphism. (See Figure 1.2.) Clearly this definition is not strong enough to capture the knot equivalence we have in mind.

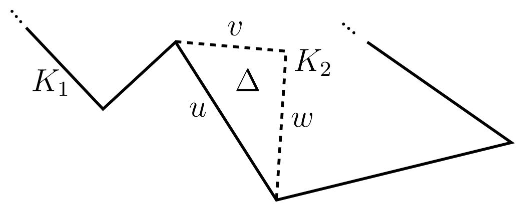

Figure 1.3: Polygonal knot K_2 is obtained from K_1 by a Δ-move. ($K_2 = (K_1 \setminus u) \cup v \cup w$ where $u \cup v \cup w$ is the boundary of a triangle.)

In addition to Definition 1.2.5, there are some other useful ways to define knot equivalence. The first is often used to define the knot type of a knot.

Definition 1.2.7. Two knots K_1 and K_2 are **equivalent** if there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$, that is $(\mathbb{R}^3, K_1)$ is homeomorphic to $(\mathbb{R}^3, K_2)$.

Ambient isotopy is certainly a slightly stronger notion of knot equivalence if the homeomorphism h is allowed to be orientation reversing. The right and left handed trefoil knots will be distinguished by ambient isotopy, but not by the homeomorphism described above. K. Reidemeister [Reid], in his classic text "Knotentheorie", defined another kind of equivalence for polygonal knots. This is often the definition found in textbooks on knot theory.

Definition 1.2.8. A **polygonal curve** is made up of a finite number of straight segments joined at the vertices. (For polygonal curves in Hadamard manifolds the segments will be made up of a finite number of geodesic segments.) A **polygonal knot** is a closed polygonal curve.

Definition 1.2.9. (Δ-move) Let K_1 be a polygonal knot in $\mathbb{R}^3$ and Δ a triangle in $\mathbb{R}^3$. Suppose $\partial\Delta = u \cup v \cup w$ and $\Delta \cap K_1 = u$. Then $(K_1 \setminus u) \cup v \cup w$ defines another polygonal knot K_2. We say K_2 results from K_1 by a Δ-move. (The reverse process is called a Δ^{-1} move.) See Figure 1.3.

Definition 1.2.10. Two polygonal knots K_1 and K_2 are **isotopic by moves** or **combinatorially equivalent** if there is a finite sequence of Δ and Δ^{-1} moves which transforms one knot into the other.

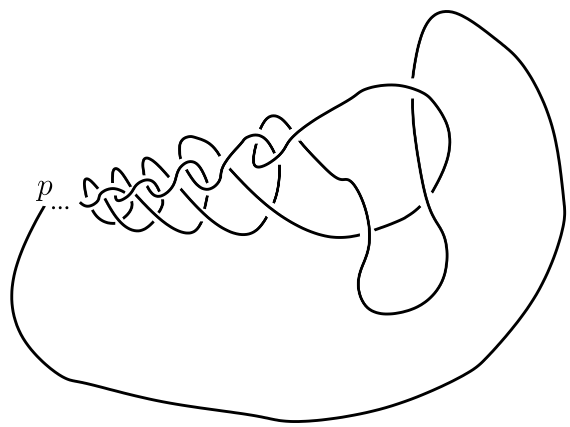

Figure 1.4: This is an example of a wild knot. It is not equivalent to the trivial knot even though finitely many "stitches" may be unraveled from the right of p.

K. Reidemeister [Reid] showed that two knots that are isotopic by moves are also ambient isotopic and vice versa. G.M. Fisher [Fish] proved that an orientation preserving homeomorphism $h : S^3 \to S^3$ is isotopic to the identity. Thus it is possible to prove that there is no difference between the apparently different definitions of knot equivalence.

Proposition 1.2.11. *The following definitions of knot equivalence are equivalent:*

1) ambient isotopy

2) orientation preserving homeomorphism

3) isotopy by moves.

Proof. See G. Burde and H. Zieschang [BZ] Ch 1. □

We will work with oriented knots and knots will be equivalent via ambient isotopies. However, the class of curves is still too large. Homeomorphisms from S^1 to $\mathbb{R}^3$ can be pathological in nature, there may be so called "wild" knots. A wild knot discovered by R. Fox [Fox] is shown in Figure 1.4. It is wild because of the behavior of the knot at the point p. Observe that finitely many "stitches" of the knot can be unraveled from the right, yet this knot is not equivalent to the trivial knot. We restrict the class of knots we examine to tame knots.

Definition 1.2.12. A knot is **tame** if it is ambient isotopic to a polygonal curve. A link is **tame** if each component is ambient isotopic to a polygonal curve.

Just as there are many equivalent definitions of knot equivalence, there are many equivalent definitions of tameness. This is due to the choice of category chosen to describe knots. Because we work in $\mathbb{R}^3$, many categories (for example, PL-category, Diff-category) are equivalent and it is a matter of choice how to describe knots. For example, in the text “Knots” by G. Burde and H. Zieschang [BZ] knots are mostly described in the PL-category. However the authors prove that any two tame knots that are topologically equivalent if and only if there are PL-representatives that are (PL) equivalent.

We are interested in actual geometric knots, so it is important to understand the exact relation between, say equivalent smooth and PL knots. Another definition of tameness helps to show this.

Definition 1.2.13. A knot is **tame** if it is ambient isotopic to a smooth curve.

This definition is equivalent to the first because of the following proposition.

Proposition 1.2.14. *Given a smooth knot, there exists an inscribed polygonal knot (arbitrarily close to it) that is ambient isotopic to it. In fact, any inscribed polygon which is sufficiently close is ambient isotopic.*

Proof. See R.H. Crowell and R.H. Fox [CF] Appendix I. □

Yet another definition of tameness involves the total curvature of the knot.

Definition 1.2.15. The **total curvature** $\kappa(P)$ of a closed polygonal curve P is the sum of the exterior angles. The **total curvature** of a closed curve C is $\sup_P\{\kappa(P)\}$ where P ranges over all polygons inscribed in C.

The theorem below shows that this definition matches the definition for smooth knots. The total curvature is the total angle through which the unit tangent vector turns — the length of the tangent indicatrix.

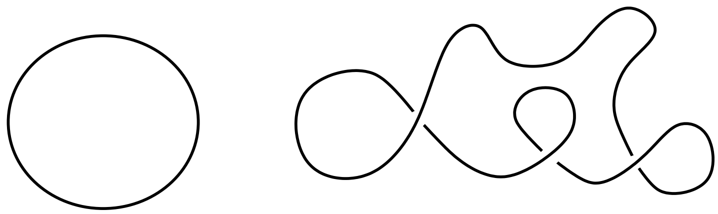

Figure 1.5: Different unknots.

Theorem 1.2.16. *Let C be a closed curve of class C^2 parameterized with respect to arclength s, then $\kappa(C) = \int_C |\gamma''(s)|\,ds$.*

Proof. See J.W. Milnor [Mil]. □

Definition 1.2.17. A knot is **tame** if it is ambient isotopic to a knot of finite total curvature.

This definition of tameness is equivalent to the other two because of the following proposition.

Proposition 1.2.18. *Given a knot of finite total curvature, there exists an inscribed polygonal knot (arbitrarily close to it) that is ambient isotopic to it.*

Proof. See S.B. Alexander and R.L.Bishop [AB], but originally due to J.W. Milnor [Mil]. □

We will use the proof of this proposition in Section 4.1. There, we develop techniques (for example Proposition 4.1.6) that describe how close two knots need to be in order to be ambient isotopic.

Definition 1.2.19. A knot K is **trivial** or called an **unknot** if it is ambient isotopic to a planar circle (standard S^1). Equivalently, K is trivial if there is an embedded disk D with $\partial D = K$ in $\mathbb{R}^3$. A m-component link is **trivial** if it bounds a collection of m disjoint (embedded) disks in $\mathbb{R}^3$.

Some examples of trivial knots can be found in Figure 1.5. The two definitions of the unknot are hard to use in practice. It is hard to construct an ambient isotopy or an embedding of a disk. We need a practical criterion of triviality for a knot in $\mathbb{R}^3$. Let f be a map from a disk D to $\mathbb{R}^3$ such that $f(\partial D) = K$. This disk is not necessarily embedded. The following lemma gives a criterion to tell when it may be replaced by an embedded disk. Let the set of self-intersections of the disk be denoted by $S(f) = \{x \in D | \, \exists y \, : y \neq x \,\text{and}\, f(x) = f(y)\}$.

Lemma 1.2.20 (Dehn). *Let f be a piecewise linear map from the disk D to $\mathbb{R}^3$, such that*

$$\overline{S(f)} \cap \partial D = \emptyset \, .$$

Then there exists an embedding $g : D \to \mathbb{R}^3$ such that $g|_{\partial D} = f|_{\partial D}$.

Proof. See D. Rolfsen [Rolf] Appendix B. □

Dehn's Lemma is one of the basic results of three dimensional topology. It was announced by M. Dehn [Dehn] in 1910, but an error in the proof was found by H. Kneser [Kne] in 1929. It was not until 1957 that C.D. Papakyriakopoulos [Pap] proved the lemma. The main condition of the lemma states that self-intersections of the disk must occur a finite distance "away from" the boundary of the disk. When this happens then the disk may be replaced by an embedded one, giving $K = \partial D$ the unknot. Thus, to show a knot K is trivial, it is sufficient to show it has a spanning disk whose interior is not intersected by K. This is easier to show in practice that finding an actual embedded disk bounded by the knot. The proof of the existence of alternating quadrisecants will be in essence a proof by contradiction using Dehn's Lemma. The absence of alternating quadrisecants will allow the construction of a disk as in Dehn's Lemma.

Throughout the rest of this work, a knotted curve will denote an oriented nontrivial tame knot in $\mathbb{R}^3$.

We now define several knot and link invariants which will be used in the rest of this work. We first discuss several invariants that are defined on the projection of a link onto a plane. A link may be projected to the plane $p : \mathbb{R}^3 \to \mathbb{R}^2$ so that the projection has nice properties. More formally,

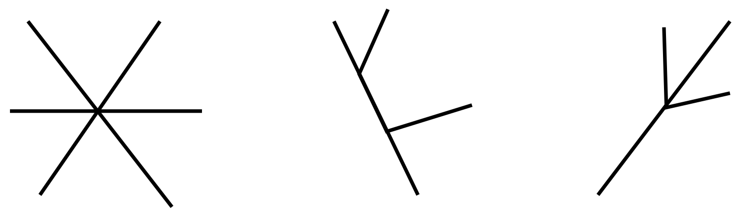

Figure 1.6: Details of projections of a polygonal knot that are not regular as in Definition 1.2.21. The left picture breaks condition 2 as three line segments project to one point. The middle picture breaks condition 3 as two disjoint line segments project to an interval. The right picture breaks condition 3 as an endpoint of a line segment projects to a disjoint line segment.

Definition 1.2.21. A **regular projection** of a polygonal knot is a projection such that

1) each line segment projects to a line segment in $\mathbb{R}^2$

2) no point belongs to the projection of 3 segments

3) projections of two segments intersect in at most one point. For disjoint segments, this point is not an end point.

Figure 1.6 illustrates non-regular projections. This definition may be generalized to non-PL links. It turns out that nearly all projections are regular.

Proposition 1.2.22. *A polygonal knot is equivalent (under rotation) to a polygonal knot in regular position. Hence regular projections are dense in the set of all projections.*

Proof. See R.H. Crowell and R.H.Fox [CF]. □

Definition 1.2.23. The regular projection of a knot K is called a **knot diagram** D. (For example, Figure 1.1 shows a knot diagram of the trefoil and figure eight knots.) An **arc** is a continuous segment in a diagram. Arcs meet at **crossings**. An unbroken arc at a crossing is an **overpass**, the remaining two arcs form an **underpass**.

There is a basic theorem by Reidemeister [Reid] which shows that the diagrams of two equivalent links are connected by a series of Reidemeister moves. The three Reidemeister moves are illustrated in Figure 1.7.

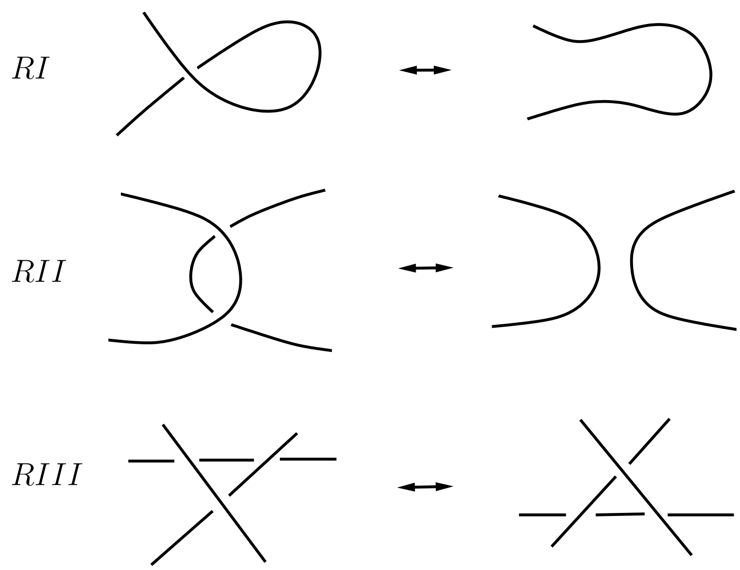

Figure 1.7: The three Reidemeister moves captures the changes in the knot diagram as either the knot changes or the direction of projection changes. RI corresponds to a strand of the knot twisting, RII corresponds to two strands overlapping, RIII corresponds to one strand passing under the crossing of two other strands.

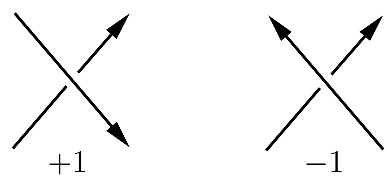

Figure 1.8: Sign of different crossings.

Theorem 1.2.24. *Two knots are ambient isotopic if and only if their diagrams are connected by a finite number of Reidemeister moves.*

Proof. Originally due to K. Reidemeister [Reid], but K. Murasugi [Mur] also gives an excellent proof. □

Definition 1.2.25. An **orientation** on a link diagram is a choice of direction on each component, usually indicated by arrows.

A crossing in a diagram of an oriented link can be allocated a sign. Crossings are labeled positive (+1) or negative (−1) as illustrated in Figure 1.8. The crossing signs may be easily remembered. Place the right thumb in the direction of the underpass. The fingers will show the sign of the overpass according to the usual right hand rule. Note that for a knot, the

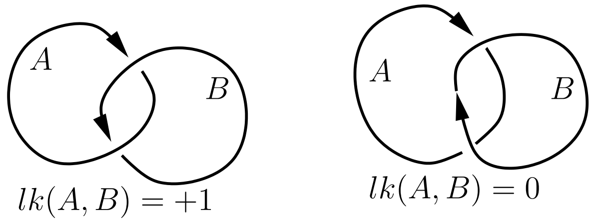

Figure 1.9: Examples of linking number for the Hopf link (left) and the unlink (right).

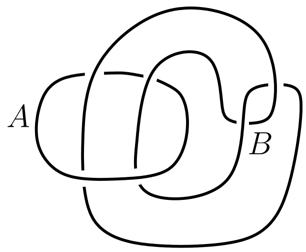

Figure 1.10: The Whitehead link has linking number 0, but is still linked.

sign of the crossing does not depend on the orientation chosen. Reversing the orientations of both strands at a crossing leaves the sign unchanged.

Definition 1.2.26. Given an oriented link diagram D, with components $C_1, \ldots, C_m$, define the **linking number of** C_i **with** C_j $(i \neq j)$ to be one half the sum of the signs of crossings of C_i with C_j. This is denoted $lk(C_i, C_j)$. The **linking number of** D is the sum of all linking numbers of all pairs of components: $lk(D) = \sum_{1 \leq i < j \leq m} lk(C_i, C_j)$.

Linking number is well defined. Any two diagrams for a link L are related by a sequence of Reidemeister moves and the linking number is not changed by such moves. The definition of linking number is also symmetric: $lk(A, B) = lk(B, A)$.

Figure 1.9 gives two examples of linking number. The left picture shows the Hopf link with $lk(A, B) = \frac{+1+1}{2} = +1$ and the right picture shows the two component unlink with $lk(A, B) = \frac{+1-1}{2} = 0$. One might expect that if two simple closed curves A and B are linked, then the linking number is never zero. This turns out not to be the case. Figure 1.10 illustrates the Whitehead link which has linking number 0, but is still linked.

E. Pannwitz [Pann] explored the different ways two simple closed curves A and B can be linked. I'll repeat (and modernize) her discussion as these different notions of linking will be used extensively in the definition of essential secants found in Definition 3.1.2 and Definition 3.1.3. Recall two simple closed curves A and B are linked if they do not bound 2 disjoint embedded disks. The homology and homotopy classes of one component (say B) in the complement of the other (say $\mathbb{R}^3 \setminus A$) give some measure of being linked. It turns out that neither class is as strong as the definition of being linked.

Linking numbers embody some elementary homology theory. Suppose K is a knot in S^3, then K has a regular neighborhood N that is a solid torus. Let the **exterior** X of K be the closure of $S^3 \setminus N$. Then X is a connected 3-manifold and its boundary ∂X is a torus. Now X has the same homotopy type as $S^3 \setminus K$ and $X \cap N = \partial X = \partial N$ and $X \cup N = S^3$. The following theorem describes the relationship between linking number and homology classes in X.

Theorem 1.2.27. *Let K be an oriented knot in (oriented) S^3 and let X be the exterior of K as above. Then $H_1(X)$ is canonically isomorphic to $\mathbb{Z}$ and is generated by the class of a simple closed curve μ in ∂N that bounds a disk in N meeting K at one point. If γ is an oriented simple closed curve in X, then the homology class $[\gamma] \in H_1(X)$ is $lk(\gamma, K)$.*

Proof. See W.B.R. Lickorish [Lick] Ch 1. □

Thus $lk(\gamma, K) = 0$ if and only if γ is null-homologous in $\mathbb{R}^3 \setminus K$ (or $S^3 \setminus K$). Also note that if $lk(\gamma, K) \neq 0$ then γ and K are linked according to the definition.

When deciding whether two curves are linked or not, it is also common to look at the homotopy class of curves in the complement of the knot. Let the **Knot Group** G denote the fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$ of the complement of K in $\mathbb{R}^3$ (or S^3).

One may wonder if curves which are null-homologous in $\mathbb{R}^3 \setminus K$ are also null-homotopic in $\mathbb{R}^3 \setminus K$. The answer is clearly no. Figure 1.11 illustrates an unknotted curve B in $\mathbb{R}^3 \setminus A$ where A is the Trefoil knot. Curve B represents an element of the commutator of the knot group of A, so $lk(A, B) = 0$. Thus B is null-homologous in $\mathbb{R}^3 \setminus A$ by definition, but it is not null-homotopic. Now as B is the unknot, the homology and homotopy groups of $\mathbb{R}^3 \setminus B$ are

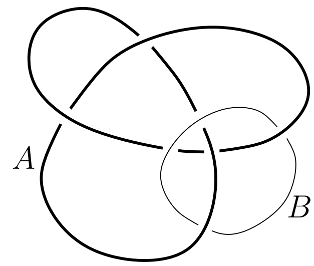

Figure 1.11: Curve B is null-homologous, but not null-homotopic, in $\mathbb{R}^3 \smallsetminus A$. Curve A is both null-homotopic and null-homologous in $\mathbb{R}^3 \smallsetminus B$.

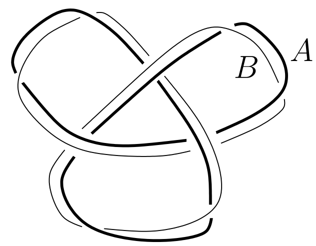

Figure 1.12: Two knots each null-homologous, but not null-homotopic, in the complement of the other.

equal. Thus A is both null-homotopic and null-homologous in $\mathbb{R}^3 \smallsetminus B$. Thus being linked in homotopy is not a symmetric relation and is a stronger notion than being linked in homology.

Figure 1.12 shows an example of a trefoil knot A and a curve B which is parallel to it. (That is A and B cobound an embedded annulus.) Curve B has been chosen so that it has linking number 0 with K (is null-homologous). However, B is not null-homotopic in $\mathbb{R}^3 \smallsetminus A$. If it were, use the homotopy to create an embedded disk with B as its boundary. Then glue in the embedded annulus. This gives an embedded disk with A as its boundary, a contradiction to the knottedness of A. Similarly, A is not null-homotopic in $\mathbb{R}^3 \smallsetminus B$. This example is different to the Whitehead link example where the components are both null-homologous and null-homotopic in the complement of the other.

Finally, we define some classes of loops in $\mathbb{R}^3 \smallsetminus K$ which occur so frequently that they have their own names.

Definition 1.2.28. Let K be an oriented knot in S^3 (or $\mathbb{R}^3$) with solid torus neighborhood N. A **meridian** μ of K is a non-separating simple closed curve that bounds a disk in N. A

longitude λ of K is a simple closed curve in ∂N that is homologous to K in N and null-homologous in the exterior of K. Clearly, $lk(\mu, K) = +1$, $lk(\lambda, K) = 0$ and $lk(\mu, \lambda) = +1$.

Chapter 2

Secants and Trisecants

2.1 Secants, trisecants and quadrisecants

Recall that a **knotted curve** denotes an oriented nontrivial tame knot in $\mathbb{R}^3$. (By knot, we mean a homeomorphic image of S^1 in $\mathbb{R}^3$, modulo reparametrizations. By tame, we mean the knot is ambient isotopic to a polygonal knot.)

Definition 2.1.1. An n**-secant line** for a knot K is an oriented line in $\mathbb{R}^3$ whose intersection with K has at least n components.

Definition 2.1.2. An n**-secant** is an ordered n-tuple of distinct points in K (no two of which lie in a common straight subarc of K) which lie in order on an n-secant line. A 2-secant is called a secant, a 3-secant is called a trisecant, a 4-secant is called a quadrisecant and a 5-secant is called a quintisecant.

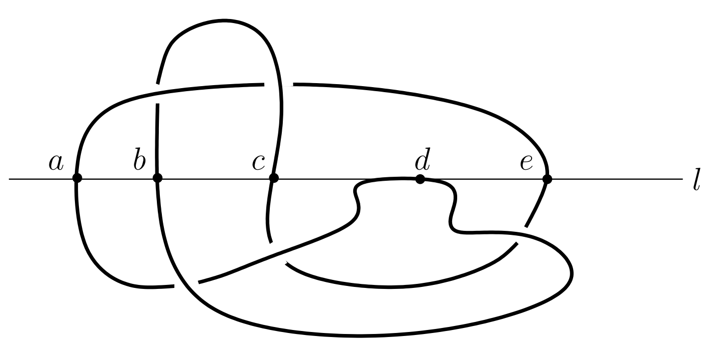

Figure 2.1: A quintisecant line l and quintisecant $abcde$.

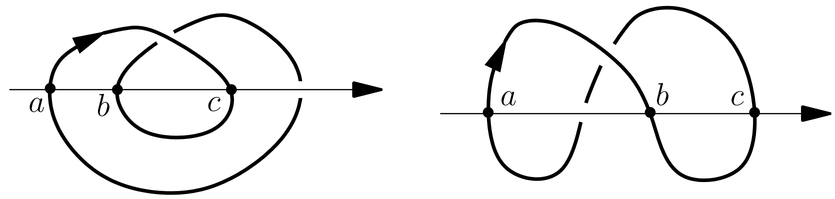

Figure 2.2: From left to right, trisecants of different and same ordering. The points of each trisecant have order *abc* along the line. The left has different order *acb* along the knot and the right has same order *abc* along the knot.

The line through an n-secant may intersect the knot in more than n places, but this does not affect the definition of an n-secant. Figure 2.1 shows a knot with quintisecant line l and quintisecant *abcde*. For example, the quintisecant line l also includes quadrisecant *abcd*, trisecant *abe*, and secant *bd*.

In K^n, let $\tilde{\Delta}$ denote the set of n-tuples in which some pair of points lie in a common straight subarc of K. Also let Δ denote the big diagonal, the set of n-tuples in which some pair of points are equal. Then $\Delta \subset \tilde{\Delta}$ (as is clear for polygonal knots). From the definition, a secant is an ordered pair of distinct points of K which do not lie on the same straight subarc of K. Clearly any such pair of points determines a secant line. Thus the set of secants $S = K^2 \setminus \tilde{\Delta}$ and is topologically an annulus. More generally, a collection of n distinct points do not necessarily lie on a line, hence the set of n-secants is contained in $K^n \setminus \tilde{\Delta}$.

Consider the set of trisecants. Given trisecant *abc* whose points lie in that order along the trisecant line. There are $|S_3| = 6$ possibilities for their ordering along K. Along K, the order is just a cyclic order. Thus there are $|S_3/C_3| = 2$ cyclic orderings of the oriented knot and oriented trisecant line. Picking the lexicographically least element in each coset, the cyclic orders are *abc* and *acb*. These orderings are respectively called *same* and *different*. Figure 2.2 illustrates the two types of trisecant. In the left picture we see the intersection points along the trisecant line and along the knot have the *same* ordering *abc*. In the right picture the points have ordering *abc* along the trisecant line, but *different* order *acb* along the knot.

We may make a more general observation about the ordering of points along a knot. Let

$xyz \in K^3$ and assume that x, y and z are distinct. As K is oriented, moving from x to y to z along K will either match the orientation of K or not. These are the two cyclic orderings discussed above. Let $\mathcal{S}$ be the set of triples of K^3 where the triples have the *same* ordering as their ordering along the knot and let $\mathcal{D}$ be the set of triples of K^3 where the ordering of the triples *differs* to their ordering along the knot. Observe that $\mathcal{S}$ and $\mathcal{D}$ *are* the connected components of $K^3 \setminus \tilde{\Delta}$. We see $\mathcal{S}$ and $\mathcal{D}$ are connected as perturbing x, y and z a little will not change their ordering. The only way to change the order along the knot say from xyz to xzy is for x, y and z to no longer be distinct. That is, either $x = y$, $y = z$ or $x = z$. Let Δ denote the big diagonal in K^3, that is $\Delta = \{(x, y, z) \in K^3 \,|\, x = y \,\text{or}\, y = z \,\text{or}\, x = z\}$.

Definition 2.1.3. Let $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$ denote the set of all trisecants of a knot K. Let $\mathcal{T}^s = \mathcal{T} \cap \mathcal{S}$ denote the set of trisecants where the order of the three points along the trisecant line is the *same* as their order along the knot. Let $\mathcal{T}^d = \mathcal{T} \cap \mathcal{D}$ denote the set of all trisecants where the order of the three points along the trisecant line is *different* to their order along the knot. Clearly $\mathcal{T}^s \cap \mathcal{T}^d = \emptyset$.

Just as with trisecants, we may compare the orderings of the four points of a quadrisecant along the quadrisecant line and along the knot. Given quadrisecant *abcd* whose points lie in that order along the quadrisecant line, there are $|S_4|$ possibilities for their ordering along K. Again, the order along K is only a cyclic order and ignoring the orientation of K is just a dihedral order. Thus there are $|S_4/D_4| = 3$ dihedral orderings of an *oriented* quadrisecant and *unoriented* knot. The three equivalence classes or types of quadrisecants are represented by *abcd*, *abdc* and *acbd*, where we have chosen the lexicographically least element in each coset as the name for each.

Definition 2.1.4. Quadrisecants of type *acbd* are called **alternating quadrisecants**. Quadrisecants of types *abcd* and *abdc* are called **simple** and **flipped** respectively.

Figure 2.3 illustrates these orderings. Alternating quadrisecants (see the right-most quadrisecant in Figure 2.3) are so named as the quadrisecant's points alternate from one

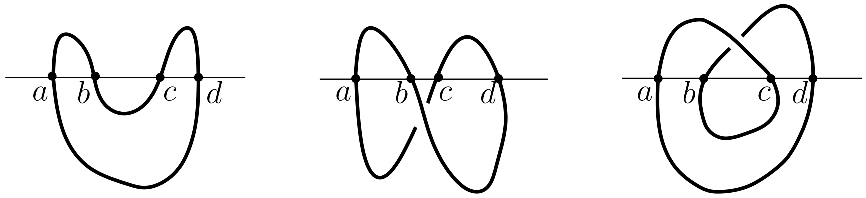

Figure 2.3: Here we see quadrisecants *abcd* on each of three knots. From left to right, these are simple, flipped and alternating, because the dihedral order of the points along K is *abcd*, *abdc* and *acbd*, respectively.

end of the quadrisecant line to the other as the ordering along the knot is followed.

In general, for an n-secant, there are $n!/n$ types of cyclic orders of an oriented n-secant and oriented knot and $n!/2n$ types of dihedral orders of an oriented n-secant and unoriented knot.

Each closed curve has a 2-parameter family of secants $S = K^2 \smallsetminus \tilde{\Delta}$. It turns out that knotted curves also have trisecants as seen in the following lemma originally due to E. Pannwitz [Pann].

Lemma 2.1.5. *Each point of a knotted curve K in $\mathbb{R}^3$ is the first point of at least one trisecant.*

Proof. Suppose there is a point p of the knot which is not the start point of any trisecant. That is, no points $q, r \in K$ are collinear with p, with p as the start point. The union of all chords $\overline{pq}$ for $q \in K$ is a disk with boundary K. This disk has no self intersections. By construction all chords intersect at p. If two chords $\overline{pq}$ and $\overline{pr}$ intersect in another place, then they overlap and one is a subinterval of another. They form a trisecant (pqr or prq), contrary to the assumption. Thus the disk is embedded and K is the unknot, contradicting the assumption of knottedness. □

From Lemma 2.1.5 we see that if we view the knotted curve K from a point p on the knot, then the trisecants occur where we see a crossing of K. Thus we expect that there will be a finite set of trisecants for any given p. Indeed, E. Pannwitz [Pann] proved for generic polygonal knots that there are at least $2u$ trisecants from any point p of K. Here u

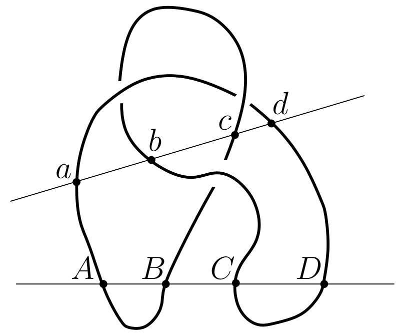

Figure 2.4: A trefoil knot with alternating quadrisecant *abcd* and simple quadrisecant $ABCD$. Other trefoil knots might not have the simple quadrisecant shown. However, the Main Theorem shows that every knotted curve has an alternating quadrisecant.

is the unknotting number of the knot defined in the Introduction. C. Schmitz [Schm] has shown Lemma 2.1.5 also holds for knotted curves in Hadamard manifolds. In Section 2.3 we will show that the set of trisecants is an embedded 1-manifold in K^3. What about higher order secants? In Proposition 2.2.8 we will see that knotted curves do not necessarily have n-secants where $n \geq 5$. However, knotted curves do have quadrisecants! Figure 2.4 shows a trefoil with two quadrisecants. Quadrisecant *abcd* is an alternating quadrisecant and quadrisecant $ABCD$ is a simple quadrisecant. The alternating quadrisecant captures the knottedness of a knotted curve. The trefoil knot, for example, does not necessarily have the simple quadrisecant shown. The rest of this work will be dedicated to proving the following theorem:

Main Theorem. *Every knotted curve in* $\mathbb{R}^3$ *has an alternating quadrisecant.*

As immediate corollaries, we find new proofs to two theorems, both of which are important results in their own right. The first theorem is the Fáry-Milnor Theorem [Mil, Far] which states that the total curvature of a knotted curve in $\mathbb{R}^3$ is greater than or equal to 4π. (In 1998 C. Schmitz [Schm] and S.B. Alexander and R. Bishop [AB] independently proved this curvature result for knotted curves in Hadamard manifolds.) The second theorem (found in [CKKS]) states that the second hull of a knotted curve in $\mathbb{R}^3$ is nonempty. The statements of these theorems and new proofs can be found in Section 4.2.

The Main Theorem has another application found in [DDS] and in Chapter 5. Recall that the ropelength of a knot is the quotient of its length and its thickness. The thickness is the radius of the largest embedded normal tube around the knot. The numerical value of the minimum ropelength needed to tie any nontrivial knot is not currently known. In joint work with J.M. Sullivan and Y.Diao [DDS], essential alternating quadrisecants are used to dramatically improve the known lower bounds of ropelength.

We wish to understand when alternating quadrisecants occur. Observe that quadrisecants come about when several trisecants share common points. Quadrisecant $abcd$ includes four trisecants (1)abc, (2)abd, (3)acd, (4)bcd. E. Pannwitz [Pann] showed the existence of quadrisecants by looking for pairs of trisecants like (1)abc and (3)acd. Here, the first and third points of trisecant abc are the same as the first and second points of trisecant acd. G. Kuperberg [Kup] showed the existence of quadrisecants by looking for pairs of trisecants like (2)abd and (3)acd. Here the first and third points of the trisecants are the same. C. Schmitz [Schm] nearly showed that quadrisecants exist by looking for pairs of trisecants like (1)abc and (2)abd, where the first and second points of the trisecants are the same. I will use Schmitz' approach, as it will allow us to use the orderings of coincident trisecants to determine the ordering of the quadrisecant.

Let quadrisecant $abcd$ be an alternating quadrisecant as in Figure 2.5. As we have seen it includes several trisecants: abc and bcd are trisecants of different ordering, and abd and acd are trisecants of same ordering. Given an alternating quadrisecant, there will be trisecants of same and different ordering which have common points. For example, $abd \in \mathcal{T}^s$ and $abc \in \mathcal{T}^d$ have the first two points in common. In fact the converse is true: if trisecants of same and different order have the first two points in common, then there is an alternating quadrisecant.

Consider $\mathcal{T}$ projected to the set of secants $S = K^2 \smallsetminus \tilde{\Delta}$ of the knot K. We project the trisecant xyz onto the secant xy, based on the first two coordinates. The map $\pi_{12} : K^3 \to K^2$ is defined by $\pi_{12}(xyz) = xy$. Let $T = \pi_{12}(\mathcal{T})$ denote the set of trisecants projected to S.

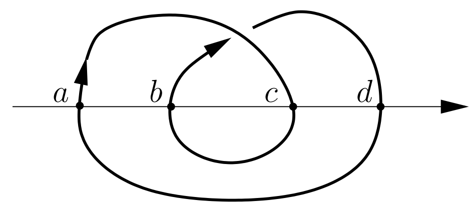

Figure 2.5: Alternating quadrisecant *abcd* includes trisecants of same order *abd* and different order *abc* which share the first two points *ab*.

Thus secant $xy \in S$ is also in T if it is the first two points of a trisecant xyz. We denote trisecants of same ordering in S by $T^s = \pi_{12}(\mathcal{T}^s)$ and trisecants of different ordering in S by $T^d = \pi_{12}(\mathcal{T}^d)$.

Lemma 2.1.6. *Let $ab \in T^s \cap T^d$ in S. This means that there exists c, d such that $abc \in \mathcal{T}^d$ and $abd \in \mathcal{T}^s$. Then either abcd or abdc is an alternating quadrisecant.*

Proof. Trisecants *abc* and *abd* lie on a common line, which must be a quadrisecant line. (Points c and d do not lie on the same straight subarc of K. If they did, then trisecants *abc* and *abd* would have the same ordering.) The ordering of the intersection points along the quadrisecant line is either *abcd* or *abdc*. Assume the order is *abcd*. Using the definition of T^s and T^d, the order of the intersection points along the knot must be *acbd*. But this means that *abcd* is an alternating quadrisecant. Now assume that the order along the quadrisecant is *abdc*. Again we deduce the order along the knot must be *acbd*, but this is equivalent to *adbc*. Hence *abdc* is also an alternating quadrisecant. □

Thus to prove that any knotted curve has at least one alternating quadrisecant, it is sufficient to prove that there are two trisecants, one of same and one of different ordering with the same first two points, that is $T^s \cap T^d \neq \emptyset$ in S. Eventually we will show that T is the image of a piecewise smooth immersion of a 1-manifold, that intersects itself in double points. To do this, we need to again restrict the kind of knots that we look at. This will put some restrictions on the behavior of T. We will look at generic polygonal knots (to be defined in the next section). We will then prove the existence of alternating quadrisecants for generic polygonal knots and use limit arguments to extend the result to all knotted curves.

2.2 Generic polygonal knots

In order to clearly understand the structure of the set of trisecants we must restrict the type of knots we examine. However, in order to use a limiting argument to show that all tame knots have an alternating quadrisecant, the types of knot examined should be dense in the set of all tame knots. We thus restrict our attention to generic polygonal knots. This section defines generic polygonal knots and looks at some implications of the definition for n-secants. It is also dedicated to proving that certain conditions are generic. The next section uses the conditions which define generic polygonal knots to help describe the structure of the set of trisecants in detail.

Definition 2.2.1. Let K be a closed polygonal curve in $\mathbb{R}^3$. Let the vertices of K be $v_1, \ldots, v_n \in \mathbb{R}^3$. Call K **non-degenerate** if no four v_i are coplanar and no three v_i are collinear.

An n-gon in $\mathbb{R}^3$ is determined by the position of its n vertices. Thus we identify the space of all n-gons with $(\mathbb{R}^3)^n$ with the usual topology. (This means that a set U of n-gons is open if for each $P \in U$, there exists an $\epsilon > 0$ such that any polygon P' obtained from P by moving all vertices by a distance less than ϵ satisfies $P' \in U$.)

Proposition 2.2.2. *Non-degeneracy is a* **generic** *property for n-gons. That is the set of non-degenerate n-gons is an open dense set in $\mathbb{R}^{3n}$.*

Proof. When $n = 3$, the third vertex cannot lie on the straight line spanned by the other two vertices. Thus the set of configurations where non-degeneracy fails has codimension 1 in the set of all configurations. For $n \geq 4$, the set of configurations where non-degeneracy fails is the union of finitely many sets where it fails for some given four vertices v_1, v_2, v_3, v_4 of the n-gon. Roughly speaking, the first three vertices can go anywhere in $\mathbb{R}^3$ (three degrees of freedom). If the fourth vertex lies on the plane spanned by the first three vertices, it will have only 2 degrees of freedom. The degenerate configurations given by these four vertices are an 11-dimensional object within the 12 dimensions of $(\mathbb{R}^3)^4$, hence the degenerate configurations are at least codimension 1 in the set of all configurations.

To see what kind of codimension 1 object is given by the degenerate configurations of four vertices, take the vectors $\vec{a} = v_2 - v_1$, $\vec{b} = v_3 - v_1$ and $\vec{c} = v_4 - v_1$. These span a tetrahedron. The vertices are degenerate if the volume of the tetrahedron is 0 or equivalently if $\det(A) = 0$, where A is the 3×3 matrix with rows given by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. This will give a cubic equation in the variables of $(\mathbb{R}^3)^n$ given by v_i. Thus each degenerate 4-tuple is one of these cubic hypersurfaces and the union of all such hypersurfaces is the set of degenerate n-gons. (Note that the situation when three vertices are collinear is a subset of the codimension 1 cubic hypersurfaces, so we don't need a special case to cover this situation. They will have even higher codimension.)

Thus the set of all degenerate configurations is a codimension 1 algebraic surface in the set of all configurations. This is closed and nowhere dense. Hence the non-degenerate n-gons are open and dense in $\mathbb{R}^{3n}$. □

Non-degenerate polygons have many interesting properties. Before describing these, we define some useful terms for describing the vertices and edges of polygons.

Definition 2.2.3. An edge e_k denotes a **closed** edge, including its endpoints. One vertex is **consecutive** to another vertex if they share a common edge. An edge is **adjacent** to another edge if they share a common vertex. Given a vertex v, a **neighboring edge of** v, is an edge which is adjacent to one of the edges that v belongs to. (See Figure 2.7.)

Proposition 2.2.4. *Any non-degenerate polygon K has the following properties:*

(1) It is embedded.

(2) The line connecting any two consecutive vertices only intersects K in their common edge. The line connecting any other two vertices does not intersect K again.

Proof.

(1) If K is not embedded, then K intersects itself. If two non-consecutive edges e_i, e_j intersect, then the four vertices of e_i and e_j are coplanar, contradicting non-degeneracy.

(2) Any two consecutive vertices share a common edge e_k which determines a line E_k. Suppose E_k intersects K in another edge e_l. Then the vertices of e_k and e_l are coplanar, contradicting non-degeneracy. Now suppose the line connecting any two (non consecutive) vertices v_i and v_j intersects K in edge e_l. Then v_i, v_j and the vertices of e_l are coplanar, again contradicting non-degeneracy. □

These three conditions provide information about non-degenerate polygons, as well as secant lines, trisecant lines and higher order secant lines. Condition (1) tells us that a non-degenerate polygon is indeed a knot. Condition (2) tells us that each component of intersection of an n-secant line with a non-degenerate polygon K is a single point. Also no two adjacent edges of K are collinear. Condition (2) also tells us that there are no multi-vertex trisecant lines. Trisecant lines (and higher order secant lines) can intersect K in at most **one** vertex. There are three more properties of knots and n-secants which may be deduced from non-degeneracy.

(3) Two coplanar edges will always have a common vertex, else the plane will contain four vertices of the non-degenerate polygon. In other words, the lines determined by any two non-consecutive edges of the non-degenerate polygon are skew.

(4) A line intersecting the interiors of two adjacent edges does not hit any vertex of K. If it did then the plane spanned by the adjacent edges also contains the vertex.

(5) The three points of a trisecant do not lie on consecutive edges. Suppose the three points of trisecant t lie on consecutive edges e_1, e_2 and e_3. Then e_1 and e_2 have a vertex in common, as do e_2 and e_3. Therefore e_1 and e_3 lie in the plane determined by t and e_2 and thus have vertex in common. But the three edges cannot form a triangle as they are intersected by t in three distinct points.

Condition (2) leads to a definition.

Definition 2.2.5. A **vertex trisecant (quadrisecant)** of a polygonal knot K is a trisecant (quadrisecant) which includes one vertex of K.

Recall that a doubly-ruled surface is a surface with two rulings of straight lines. Every line of one ruling intersects every line of the other ruling (maybe at infinity) and each point on the doubly-ruled surface lies on exactly one line from each ruling. It is a well known fact that any three pairwise skew lines generate a doubly-ruled surface.

Proposition 2.2.6. *A triple of pairwise skew lines E_1, E_2, E_3 determines a doubly-ruled surface, either a one-sheeted hyperboloid or a hyperbolic paraboloid. Moreover, the lines E_i will lie on one ruling of the surface and any line t intersecting all the E_i will lie on the other ruling. There is an open interval or a circle of such lines t.*

Proof. The proof may be found in [PW] Ch 3 or [Otal]. We also give a self-contained proof at the end of the chapter. □

Definition 2.2.7. A non-degenerate polygonal knot in $\mathbb{R}^3$ is **generic** if it satisfies the following **genericity conditions**:

(G1) Given any three pairwise skew edges e_1, e_2, e_3 of K and the doubly-ruled surface H that they generate, no vertex of K (except endpoints of e_1, e_2, e_3) may lie on H. Also no fourth edge of K may be tangent to H.

(G2) There are no quintisecants (or higher order secants).

(G3) There are no vertex trisecant lines which lie in the osculating plane of the vertex. (The osculating plane is the plane spanned by the two incident edges.) See Figure 2.6.

(G4) There are no quadrisecant lines which intersect a vertex and one of its neighboring edges. See Figure 2.7.

Proposition 2.2.8. *The set of all generic n-gons is open and dense in $\mathbb{R}^{3n}$.*

The proof is technical and maybe omitted on a first reading. However, the reader should note that these conditions are used when describing the behavior of T in S. Indeed, condition (G1) is used in Lemma 2.3.8 and Lemma 2.3.11. Condition (G2) is used in Lemma 2.3.10, condition (G3) in Lemma 2.3.11 and Lemma 2.3.16 and condition (G4) in Lemma 2.3.11. These conditions are, in some sense, natural conditions to consider. If they are broken then the lines determined by the edges of the knot lie in degenerate configurations, see [BELSW].

Proof. We prove that n-gons with each condition form an open dense subset of $\mathbb{R}^{3n}$. We do this by showing that the sets which do *not* to satisfy each condition lie in some real algebraic variety of codimension 1 (or higher). These sets have empty interior in $\mathbb{R}^{3n}$, hence n-gons which do satisfy the conditions are open and dense in $\mathbb{R}^{3n}$. We start by assuming we have non-degenerate n-gons. This will not affect our argument, as by elementary topology we know that if Y is open and dense in Z and X is open and dense in Y, then X is open and dense in Z. Similarly, while condition (G1) is used in the proof of genericity of condition (G2), we do not have to consider the interaction between the other conditions. This is because non-degenerate polygonal knots having each condition form an open dense subset of $\mathbb{R}^{3n}$ and the intersection of two open dense subsets is still open and dense. These results about dense subsets maybe found in any standard text on topology such as [Dug].

Condition (G1): The set of configurations where condition (G1) fails is the union of finitely many sets where it fails for four given edges e_1, e_2, e_3, e_4 of n-gon K. Without loss of generality assume the lines determined by e_1, e_2 and e_3 are pairwise skew. For any given positions of e_1, e_2 and e_3, a doubly-ruled surface H is generated. Consider a fourth edge e_4 of K. Suppose one vertex lies on H, then it has 2 degrees of freedom. Hence this illegal position for e_4 is 5-dimensional (out of a possible 6 dimensions). Thus the set of illegal configurations has codimension 1. Suppose both vertices of e_4 lie on H (which must happen if e_4 is adjacent to one of the e_i). Then both vertices have 2 degrees of freedom and the set of illegal configurations again must have codimension at least 1. In either case, the n-gons satisfying condition (G1) are open and dense in $\mathbb{R}^{3n}$.

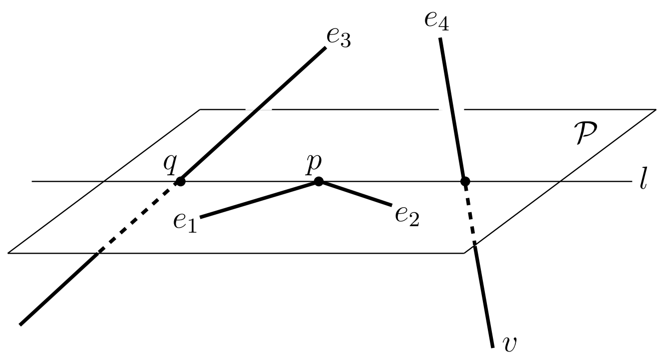

Figure 2.6: A non-generic trisecant line l lying in the osculating plane of vertex p, breaking genericity condition (G3) of the definition of a generic knot.

Now suppose a fourth edge of K is tangent to H. This will give rise to a codimension 1 condition in the set of all configurations. One vertex v of e_4 either does or does not lie on H, depending on whether e_4 is or is not adjacent to one of the e_i. Regardless of this, the second vertex may go anywhere but H and a 2-dimensional surface determined by v and H. This surface is the union of tangent lines from v to H. If v lies on H, then the lines are tangent to H at a place other than v. In either case, the illegal configurations are codimension 1 (or higher). Hence the n-gons obeying the condition are open and dense in $\mathbb{R}^{3n}$.

Condition (G2): The set of configurations where condition (G2) fails is the union of finitely many sets where it fails for five given edges e_1, e_2, e_3, e_4 and e_5 of K. For any position of e_1, e_2, e_3 and e_4, there is only some set of positions for e_5 of positive codimension which is ruled out. If there is no quadrisecant through e_1, e_2, e_3 and e_4, then there is no condition on e_5. If there is a quadrisecant through e_1, e_2, e_3 and e_4, then there will be one or two (use genericity condition (G1) and see Lemma 2.3.8 and Remark 2.3.9). The fifth edge just needs to avoid the quadrisecant line(s) in space. Thus the first vertex v of e_5 may go anywhere but the second vertex may not lie in the plane(s) spanned by the quadrisecant line(s) and vertex v. Thus a 5 dimensional surface (within $(\mathbb{R}^3)^2$ positions for e_5) is eliminated. Thus the set violating condition (G2) is codimension 1 in the set of all configurations. Hence n-gons satisfying condition (G2) are open and dense in $\mathbb{R}^{3n}$.

Condition (G3): Figure 2.6 illustrates a trisecant which fails to meet condition (G3)

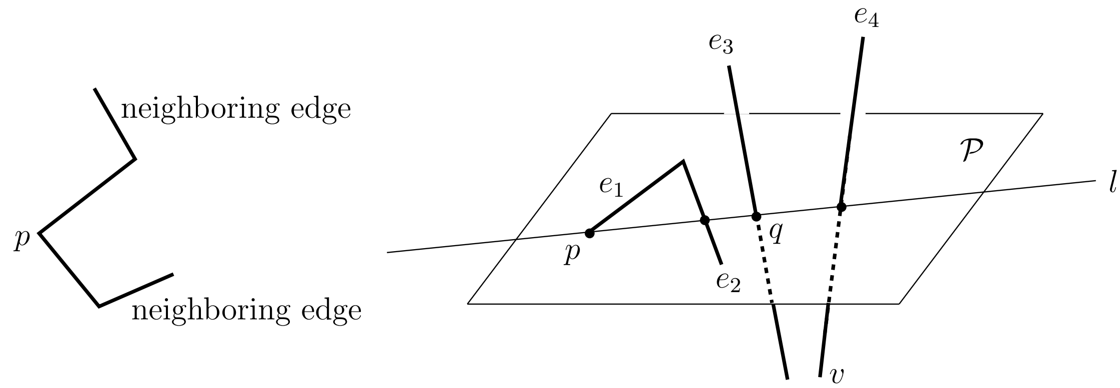

Figure 2.7: To the left, a vertex p and its neighboring edges. To the right, a non-generic quadrisecant line intersecting vertex p and its neighboring edge e_2, breaking genericity condition (G4) of the definition of a generic knot.

and lies in the osculating plane of the vertex. The set of configurations where condition (G3) fails is the union of finitely many sets where it fails for four given edges e_1, e_2, e_3, e_4 of K, where two of the edges are adjacent. Without loss of generality, let e_1 and e_2 be adjacent and let their common vertex be p. Then there is only some set of positions of e_4 of positive codimension which is to be eliminated. Edges e_1 and e_2 span a plane $\mathcal{P}$. Non-degeneracy of K means that no vertex of e_3 or e_4 will lie in $\mathcal{P}$. Aside from this there is no condition on e_3. If e_3 does not intersect $\mathcal{P}$ then there is no condition on e_4. Suppose e_3 intersects the plane $\mathcal{P}$ in the point q. Then to satisfy the condition, e_4 must avoid the line l through p and q. The first vertex v of e_4 may go anywhere, but the second one must avoid the plane through l and vertex v. Thus a 5 dimensional surface (within $(\mathbb{R}^3)^2$ positions for e_4) is eliminated, which is codimension 1 in the set of all configurations. Hence n-gons satisfying condition (G3) are open and dense in $\mathbb{R}^{3n}$.

Condition (G4): Figure 2.7 illustrates a quadrisecant which fails to meet condition (G4). The set of configurations where condition (G4) fails is the union of finitely many sets where it fails for four given edges e_1, e_2, e_3, e_4 of K, where two of the edges are adjacent. Without loss of generality, let vertex p belong to edge e_1 with edge e_2 the neighboring edge. Then there is only some set of positions of e_4 of positive codimension which is to be eliminated. Edges e_1 and e_2 span a plane $\mathcal{P}$. Non-degeneracy of K means that no vertex of e_3 or e_4

will lie in $\mathcal{P}$. Aside from this there is no condition on e_3. If e_3 does not intersect $\mathcal{P}$ then there is no condition on e_4. Suppose e_3 intersects the plane $\mathcal{P}$ in the point q. Then to satisfy the condition, e_4 must avoid the line l through p and q. The first vertex v of e_4 may go anywhere, but the second one must avoid the plane through l and vertex v. Thus a 5 dimensional surface (within $(\mathbb{R}^3)^2$ positions for e_4) is eliminated, which is codimension 1 in the set of all configurations. Hence n-gons satisfying condition (G4) are open and dense in $\mathbb{R}^{3n}$. □

We now return to doubly-ruled surfaces and Proposition 2.2.6. A proof of this result may be found in [PW] Ch 3. Here we give a self-contained proof of this result following [Otal]. The details in this proof reveal some of the abstract structure of the set of trisecants $\mathcal{T} \in K^3$ which will be used in Lemma 2.3.2 and Proposition 2.3.4.

Proposition. *A triple of pairwise skew lines E_1, E_2, E_3 determines a doubly-ruled surface, either a one-sheeted hyperboloid or a hyperbolic paraboloid. Moreover, the lines E_i will lie on one ruling of the surface and any line t intersecting all the E_i will lie on the other ruling. There is an open interval or a circle of such lines t.*

Proof. There is an affine quadratic form in $\mathbb{R}^3$ whose zero set contains the three lines E_i. To see this, identify the affine space $\mathbb{R}^3$ to the hyperplane $\{w = 1\}$ in $\mathbb{R}^4$, then homogenize the quadratic to have four variables (x, y, z, w). Note that the space of quadratic forms is 10 dimensional as there are $10 = 4 + \binom{4}{2}$ coefficients in the quadratic. For each of the lines E_i, pick three points on the line. These must all lie in the zero set of the quadratic. Thus we get $9(= 3 \times 3)$ linear equations in the coefficients. So there will be at least one non-zero quadratic form Q on $\mathbb{R}^4$ which is zero on the nine points. Let q be the restriction of Q to the hyperplane $\{w = 1\}$, q is an affine quadratic in $\mathbb{R}^3$.

In order to be part of a ruling, the whole line E_i should belong to the surface. For each line, substitute a parametric form for the line into the quadratic q. This will give a quadratic (in the parameter) which has three zeros and hence is identically zero on the line. Thus the

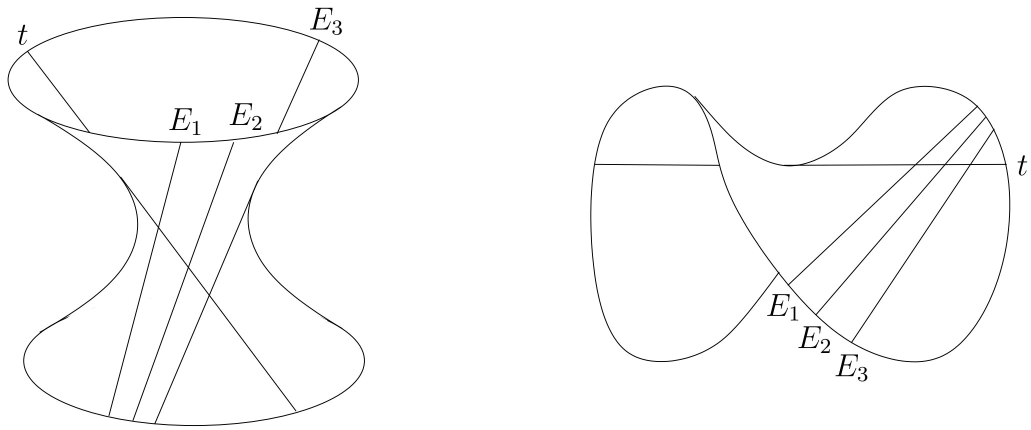

Figure 2.8: Doubly-ruled surfaces generated by three pairwise skew lines. The lines lie in one ruling and trisecants lines in the other.

whole line belongs to the surface. Now consider a line t intersecting the three lines E_i. It will intersect each E_i in one point which is on the zero set of the quadratic. As before, the whole of the line t lies in the surface and t is contained in the quadratic $q^{-1}(0)$.

Since the lines E_i are pairwise skew, the quadratic form Q in $\mathbb{R}^4$ is non-degenerate and its signature is $(2, 2)$. To understand the claim about the signature, consider what happens if the the signature is $(4, 0)$ or $(0, 4)$. Up to multiplication by -1 the quadratic is equivalent to the form $x^2 + y^2 + z^2 + w^2 = 0$, but this has no solutions when restricted to the hyperplane $\{w = 1\}$. If the signature is $(3, 1)$ or $(1, 3)$, then one of the coefficients of the quadratic is of opposite sign to the rest. Up to multiplication by -1, the quadratic is equivalent to either $x^2 + y^2 + z^2 - w^2 = 0$ or $x^2 + y^2 - z^2 + w^2 = 0$. When restricted to the hyperplane $\{w = 1\}$, the first is a sphere and the second is a 2-sheeted hyperboloid. Neither of these surfaces can have lines entirely contained in them.

Thus signature must be $(2, 2)$, and two of the coefficients of the quadratic have opposite sign. Up to multiplication by -1, the quadratic is equivalent to either $x^2 + y^2 - z^2 - w^2 = 0$ or $x^2 - y^2 + z^2 - w^2 = 0$. When restricted to the hyperplane $\{w = 1\}$ the first surface is a one-sheeted hyperboloid. The second quadratic is equivalent to $x^2 - y^2 - zw = 0$, so that when restricted to the hyperplane $\{w = 1\}$ the surface is a hyperbolic paraboloid.

For a quadratic form of this type (non-degenerate of signature (2,2)), the union of homogeneous planes in $\mathbb{R}^4$ decomposes naturally into two disjoint families with the following properties (see [Ber]):

(1) Each homogeneous line in $\mathbb{R}^4$ is contained in exactly one plane from each family.

(2) Two homogeneous planes (one from each family) have one nontrivial intersection.

The affine lines in $q^{-1}(0)$ are precisely these homogeneous planes when restricted to $\{w = 1\}$. By considering the two properties of the homogeneous planes when restricted to $\{w = 1\}$, we see that there are two families of affine lines in $q^{-1}(0)$ and each family gives one ruling of $q^{-1}(0)$.

Property (1) means that each point on the doubly-ruled surface lies on exactly one line from each ruling. Property (2) means that each line in one ruling intersects every line of the other ruling (maybe at infinity). This implies that for each point on E_i, there is one line which intersects E_1, E_2 and E_3. Thus, there is an open interval or a circle of lines intersecting E_1, E_2 and E_3. (Projectively it is always a circle). See Figure 2.8. □

2.3 The structure of the set of trisecants

The assumption of generic conditions has strong implications for the structure of the set of trisecants $\mathcal{T}$ in K^3 and in the set of secants S. **Unless stated otherwise, all the results of this section will apply to generic polygonal knotted curves.** Let $\overline{\mathcal{T}}$ denote the closure of $\mathcal{T} \in K^3$ and let the boundary of $\mathcal{T}$ be defined as $\partial\mathcal{T} := \overline{\mathcal{T}} \setminus \mathcal{T}$.

Recall that: (1) No two adjacent edges of K are collinear and (2) trisecants have three distinct points of K, no two of which lie on an edge of K. Thus $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$, where $\tilde{\Delta}$ is the set of $xyz \in K^3$ such that at least two points lie on a common edge of K. Thus $\tilde{\Delta}$ is really a union of "cubes", or more formally, $\tilde{\Delta} = \bigcup_i (e_i \times e_i \times K) \cup \bigcup_i (e_i \times K \times e_i) \cup \bigcup_i (K \times e_i \times e_i)$. We are also interested in the big diagonal: the planes in K^3 where two points of K are equal. Recall this is defined as $\Delta := \{(x, y, z) \in K^3 \,|\, x = y \,\text{or}\, y = z \,\text{or}\, x = z\}$. In a similar way, we consider the set of secants $S = K^2 \setminus \tilde{\Delta}$, a topological annulus. As K is a generic polygonal knot, $\tilde{\Delta}$ is the set of $xy \in K^2$ where both points lie on a common edge e_i. Thus $\tilde{\Delta}$ is a union of "squares", or more formally, $\tilde{\Delta} = \bigcup_i (e_i \times e_i)$. In a similar fashion we may think of K^3 as being a union of "cubes" $\bigcup_{i,j,k}(e_i \times e_j \times e_k)$.

We first examine the structure of the set of trisecants in K^3. The points of a trisecant $t \in \mathcal{T}$ lie in either

a) the interiors of the three edges e_1, e_2 and e_3, or

b) the interiors of the two edges e_1, e_2 and a vertex v.

We examine the structure of $\mathcal{T} \cap (e_1 \times e_2 \times e_3)$ in Lemma 2.3.1 and Lemma 2.3.2. A vertex of K belongs to two edges of K and so vertex trisecants are used to complete the description of $\mathcal{T}$ (and $\overline{\mathcal{T}}$) in K^3 in Proposition 2.3.4.

Assume that the three points of trisecant $t \in \mathcal{T}$ lies in the interior of three edges e_1, e_2 and e_3 of K. Let E_1, E_2 and E_3 be the lines determined by e_1, e_2 and e_3. There are two cases considered in the following lemmas. Either the E_i are pairwise skew or two of the E_i are adjacent and the third is skew to both of these.

Lemma 2.3.1. *Let t be a trisecant of a generic polygonal knotted curve K, whose points lie on edges e_1, e_2 and e_3 of K. Assume that e_1 and e_2 are adjacent edges of K, and the line determined by e_3 is skew to each of the lines determined by e_1 and e_2. Then the set of trisecants through e_1, e_2, e_3 is homeomorphic to either $[0,1]$ or $[0,1)$. Moreover, the interval of trisecants in $e_1 \times e_2 \times e_3$ lies in the plane $e_1 \times e_2 \times \{p\}$, and along the interval of trisecants each of the other two points varies smoothly and monotonically along the corresponding edges. The interval of trisecants either starts and ends in a face of $e_1 \times e_2 \times e_3$ or starts in a face and ends at vvp where v is the common vertex of e_1 and e_2.*

Proof. Let $\mathcal{P}$ be the plane spanned by e_1 and e_2 and let p be the unique point of intersection of e_3 with $\mathcal{P}$. (Non-degeneracy implies that p is in the interior of e_3.) Figure 2.9 shows $\mathcal{P}$ separated into regions divided by lines E_1, E_2 and the line l between the non-intersecting vertices of e_1 and e_2. For trisecants to occur then point p must be in one of regions 1, 2 or 3

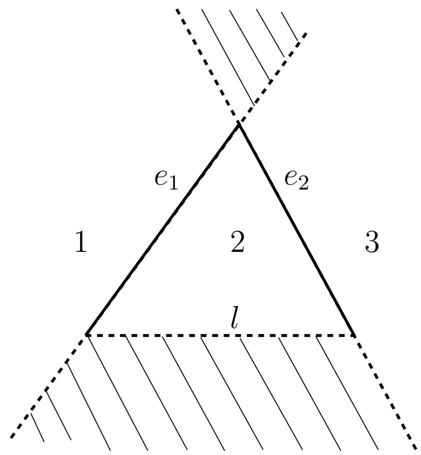

Figure 2.9: Regions in the plane $\mathcal{P}$ spanned by e_1 and e_2 determine the type of interval of trisecants. If the third edge e_3 intersects $\mathcal{P}$ in the shaded regions there are no trisecant lines intersecting e_1, e_2, e_3. If e_3 intersects $\mathcal{P}$ in region 2, the set of trisecants is homeomorphic to $[0,1]$. If e_3 intersects $\mathcal{P}$ in regions 1 or 3, the set of trisecants is homeomorphic to $[0,1)$.

(and not the shaded regions). All trisecants whose points lie on e_1 and e_2 lie in the plane $\mathcal{P}$ and must contain the point $p \in e_3$. There is a 1-parameter family of such trisecants. If p is in region 2, then the interval of trisecants will be homeomorphic to $[0,1]$ as in Figure 2.10 (left). Here trisecants are of the form xpy, where $x \in e_1$ and $y \in e_2$. If p is in either region 1 or 3, then the interval of trisecants will be homeomorphic to $[0,1)$ as in Figure 2.10 (right). Here trisecants are of the form xyp (or pxy) where $x \in e_1$ and $y \in e_2$. The set of trisecants end on the degenerate trisecant vvp (or pvv) where v is the common vertex of e_1 and e_2. This point is not in the set of trisecants, vvp (or pvv) $\in \Delta \subset \tilde{\Delta}$. Thus one end point of the interval will not be in the set of trisecants. In all these cases, the trisecants lie in a common plane and we call the set of trisecants whose points lie on e_1, e_2 and e_3 an interval of trisecants.

We may say more about the structure of $\mathcal{T} \cap (e_1 \times e_2 \times e_3)$. In all cases ($p$ in regions 1,2 or 3), the interval of trisecants lies in the plane $e_1 \times e_2 \times \{p\}$. Parameterize K with respect to arclength; then along the interval of trisecants the points move monotonically and smoothly along edges e_1 and e_2. We give an explicit calculation to show this in the Appendix. For p in region 2, we see the interval of trisecants starts at $e_1 \times \{0\} \times \{p\}$ and ends at $\{1\} \times e_2 \times \{p\}$ (or vice versa); thus it starts and ends midface on $e_1 \times e_2 \times e_3$. For p in regions 1 or 3, the

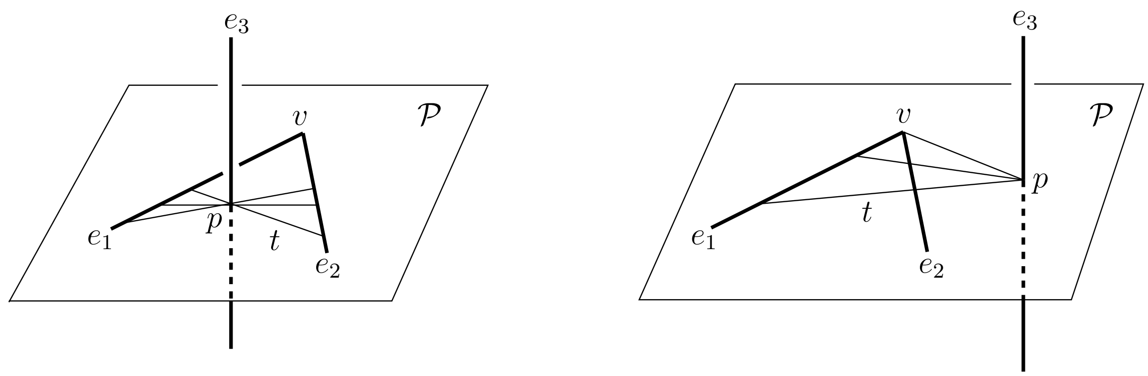

Figure 2.10: Interval of trisecants through two adjacent edges. The left picture shows an interval of trisecants homeomorphic to $[0, 1]$, the right shows an interval of trisecants homeomorphic to $[0, 1)$.

interval of trisecants start at either $e_1 \times \{0\} \times \{p\}$ or $\{0\} \times e_2 \times \{p\}$ and ends at vvp; thus it starts midface and ends on an edge of the cube $e_1 \times e_2 \times e_3$. Thus the interval of trisecants is smoothly embedded in $e_1 \times e_2 \times e_3 \subset K^3$. □

Lemma 2.3.2. *Let t be a trisecant of a generic polygonal knot K, whose points lie on edges e_1, e_2 and e_3 of K. Assuming that the lines determined by e_1, e_2, e_3 are pairwise skew, then the set of trisecants through e_1, e_2, e_3 is homeomorphic to $[0, 1]$. Moreover, this interval and starts and ends in a face of $e_1 \times e_2 \times e_3$ and along the interval of trisecants each of the three points moves smoothly and monotonically along the corresponding edges.*

Proof. By Proposition 2.2.6, the triple of pairwise skew edges will determine a doubly-ruled surface, either a one-sheeted hyperboloid or hyperbolic paraboloid. The edges e_i $(i = 1, 2, 3)$ all lie in one ruling. A trisecant line is a line in the other ruling which intersects e_1, e_2 and e_3. Let E_i be the line determined by e_i. From Proposition 2.2.6 we know there is a circle of lines intersecting E_1, E_2 and E_3. Let I_1 be the closed subinterval of this circle of lines which intersect edge e_1 within E_1. That is, I_1 represents lines which intersect the edge e_1 and the lines E_2 and E_3. As e_1 is an edge (closed), I_1 is homeomorphic to a closed interval. Let I_2 and I_3 be defined in a similar manner, they too will be homeomorphic to a closed interval. Thus the set of trisecants though e_1, e_2 and e_3 is homeomorphic to the intersection of I_1, I_2 and I_3 (three closed intervals in a common circle), which is either a closed interval or a

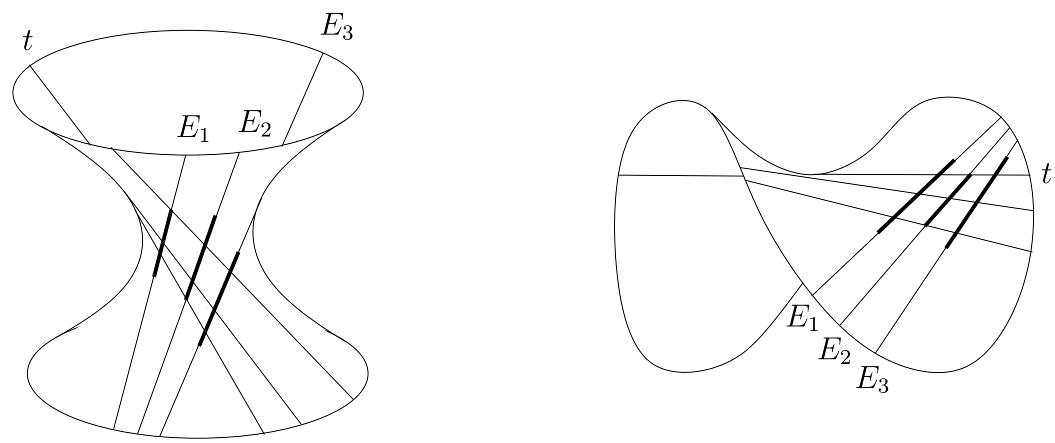

Figure 2.11: Closed interval of trisecant lines intersecting edges e_1, e_2 and e_3 on a doubly-ruled surface.

point. See Figure 2.11. Moreover, this interval of trisecants must start and end with a vertex trisecant. It certainly starts (and ends) with one of the endpoints of the I_i. The endpoints of the I_i correspond to trisecant lines with a vertex of edge e_i as a point of intersection. The interval cannot start with a line intersecting two (or more) endpoints of the I_i, as this corresponds to a trisecant with two (or more) vertices of the knot — contradicting non-degeneracy. Similarly if the interval of trisecants is a point this corresponds to a trisecant line through at least two endpoints of the I_i again contradicting non-degeneracy. Hence, if it is nonempty, the set of trisecants through e_1, e_2 and e_3 is homeomorphic to a closed interval of non-zero length which starts and ends with a vertex trisecant. We call the set of trisecants whose points lie on e_1, e_2 and e_3 an interval of trisecants.

We can say more about the structure of this interval of trisecants in $e_1 \times e_2 \times e_3$. The interval starts and ends with a vertex trisecant; thus it starts and ends in a face of $e_1 \times e_2 \times e_3$. Parameterize K with respect to arclength. From Proposition 2.2.6 the interval of trisecant lines moves monotonically along each edge. Moreover along the interval of trisecants, each of the three points moves smoothly and monotonically along the corresponding edge. This smoothness is not unexpected as the lines are varying along a quadratic surface. We give an explicit calculation to show smoothness in the Appendix. Thus $\mathcal{T} \cap (e_1 \times e_2 \times e_3)$ starts and ends on a face of $e_1 \times e_2 \times e_3$ and is smoothly embedded in $e_1 \times e_2 \times e_3 \subset K^3$. $\square$

The set of trisecants whose points lie on e_1, e_2 and e_3 and the kinds of trisecants described in Lemma 2.3.1 and Lemma 2.3.2 come up repeatedly in the work that follows, so we give them their own names.

Definition 2.3.3. We call the set of trisecants whose points lie on e_1, e_2 and e_3 an **interval of trisecants**. Let t be a trisecant whose points lie on edges e_1, e_2 and e_3 of K. If the lines determined by e_1, e_2 and e_3 are pairwise skew then we say that t is a **skew trisecant**. If two edges are adjacent and the line determined by the third edge is skew to the lines determined by the adjacent edges, then we say that t is an **adjacent trisecant**. Let the adjacent edges have common vertex v. Then adjacent trisecants lie in the **osculating plane of the vertex** v (which is the plane spanned by the two adjacent edges). A **degenerate trisecant** is a triple in K^3 of the form vvp or pvv where v is a vertex of the knot. Degenerate trisecants are contained in the big diagonal $\Delta \subset K^3$.

Thus $\mathcal{T} \cap (e_1 \times e_2 \times e_3)$ is either an interval of skew trisecants (as in Lemma 2.3.2) or an interval of adjacent trisecants (as in Lemma 2.3.1). Intervals of adjacent trisecants may end on degenerate trisecants.

Recall $\overline{\mathcal{T}}$ denotes the closure of $\mathcal{T}$ in K^3 and the boundary of $\mathcal{T}$ is defined as $\partial\mathcal{T} := \overline{\mathcal{T}} \smallsetminus \mathcal{T}$.

Proposition 2.3.4. *Let K be a generic polygonal knotted curve. In K^3, $\overline{\mathcal{T}}$ is a compact 1-manifold with boundary, embedded in K^3 in a piecewise smooth way with $\mathcal{T} \subset K^3 \smallsetminus \tilde{\Delta}$ and $\partial\mathcal{T} \subset \Delta$.*

Proof. The definition of a trisecant shows $\mathcal{T} \subset K^3 \smallsetminus \tilde{\Delta}$. Suppose pqr is a trisecant and p, q, and r are interior points of edges e_p, e_q and e_r respectively. Then the interval of trisecants that pqr belongs to is uniquely determined by e_p, e_q, e_r. Lemma 2.3.1 and Lemma 2.3.2 shows that $\mathcal{T} \cap (e_p \times e_q \times e_r)$ is a 1-manifold smoothly embedded in $e_p \times e_q \times e_r \subset K^3$. We also know these intervals of trisecants start in faces of $e_p \times e_q \times e_r$ and either they end in a face or they end at a degenerate trisecant on an edge of $e_p \times e_q \times e_r$. Take the case where an interval of trisecants ends on a face of $e_p \times e_q \times e_r$. Suppose this point is vertex trisecant vqr,

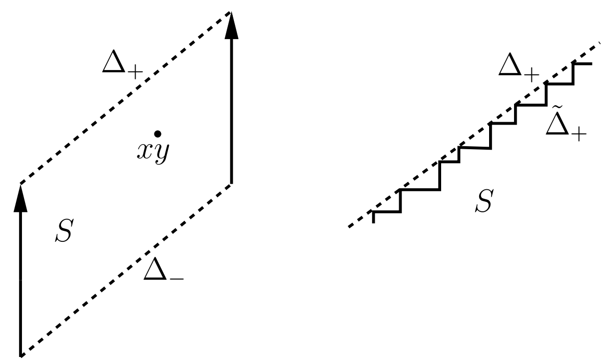

Figure 2.12: The set of secants S with path metric. On the left, the big diagonals Δ_+ and Δ_- are shown. The right picture gives the detail of Δ_+ and $\tilde{\Delta}_+$ with "steps" from edges of the polygonal knot K.

where v is a vertex of e_p and q and r lie on the interiors of e_q and e_r respectively. Now v belongs to two edges e_p and e'_p say. Thus only *one* other interval of trisecants also contains vertex trisecant vqr (the interval of trisecants in $e'_p \times e_q \times e_r$). Thus the intervals of trisecants join across the common face of the cubes via the vertex trisecant. Thus $\mathcal{T}$ is a 1-manifold embedded in K^3 in a piecewise smooth way.

To understand the claims about $\overline{\mathcal{T}}$, we must understand $\partial\mathcal{T}$. Recall that a half-open interval of trisecants occurs when an interval of adjacent trisecants ends on a degenerate trisecant (vvp or pvv where v is a vertex of K). Degenerate trisecants lie in Δ (see Lemma 2.3.1). Thus the limit point (what was added in $\overline{\mathcal{T}}$) lies in Δ, so $\partial\mathcal{T} \in \Delta$. Also note that vertex v belongs to precisely two edges of K. Thus no other interval of trisecants in K^3 can end at a degenerate trisecant (vvp or pvv). Hence $\overline{\mathcal{T}}$ is a compact 1-manifold with boundary, embedded in a piecewise smooth way in K^3. □

We now wish to understand the projection $T = \pi_{12}(\mathcal{T})$ of the set of trisecants to the set of secants $S = K^2 \setminus \tilde{\Delta}$. On S, a secant xy is a trisecant if it is the first two points of trisecant xyz.

The knot K has an orientation and is parameterized with respect to arclength. We give K^2 the product metric such that $d((x, y), (x', y')) = d(x, x') + d(y, y')$ (where $d(x, x')$ is the shorter arclength along K). Now consider S as a metric space, not with the subset metric, but with the path metric $d((x,y),(x',y')) = \inf_\gamma(len(\gamma))$, where $len(\gamma)$ is the length of γ and γ is any path in S from (x,y) to (x',y'). Let the completion of S in this metric be

$$\overline{S} = (K^2 \setminus \tilde{\Delta}) \cup \tilde{\Delta}_+ \cup \tilde{\Delta}_- \,.$$

Here $\tilde{\Delta}_-$ denotes the lower edge of S, that is approaching $\tilde{\Delta}$ in K^2 from above. That is $xy \in K^2$ and $y > x$: x and y are close together with y just after x in the order of K. Similarly $\tilde{\Delta}_+$ denotes the upper edge of S. See Figure 2.12 (right). Let Δ_-, Δ_+ denote the lower and upper big diagonals of S respectively. The topology of S is imposed by the metric. Hence the upper and lower edges of the annulus S are far from each other. The shortest path between them lies *in* S and by definition can *not* cross Δ.

Let $\overline{T}$ be the projection of $\overline{\mathcal{T}}$ in $\overline{S}$, $\overline{T} := \pi_{12}(\overline{\mathcal{T}})$. Now π_{12} maps to K^2 but $\overline{S}$ is not contained in K^2. To understand $\overline{T}$ we must describe where $\pi_{12}(\partial T)$ lies in $\overline{S}$.

A half-open interval of trisecants occurs when an interval of adjacent trisecants ends on a degenerate trisecant. See Figure 2.10 and Figure 2.13. Let the points of the adjacent trisecants lie on edges e_1, e_2 and e_3 and let e_1 and e_2 be adjacent with common vertex v. Let e_3 intersect the plane $\mathcal{P}$ spanned by e_1 and e_2 in the point p. Trisecants are ordered triples in K^3. Suppose the points of the trisecants lie on $e_1e_2e_3$ in that order, then the first two points of the trisecant both end at vertex v and the third point p is fixed. The interval of trisecants ends at degenerate trisecant $vvp \in \Delta \subset K^3$ and in $\overline{S}$, the interval ends at either Δ_+ or Δ_-. If the points of the trisecant lie on $e_3e_2e_1$ in that order, then the first point p is fixed and the second and third points both end at vertex v. This interval of trisecants ends at degenerate trisecant $pvv \in \Delta \subset K^3$ and in $\overline{S}$, it ends at pv in the interior of S.

In fact we can say more. Figure 2.13 (left) shows the knot is oriented from e_1 to e_2. Thus trisecants whose points lie on edges $e_1e_2e_3$ in that order are trisecants of same ordering T^s. Moreover, as the interval of trisecants xyp ends at degenerate trisecant vvp, the first point of

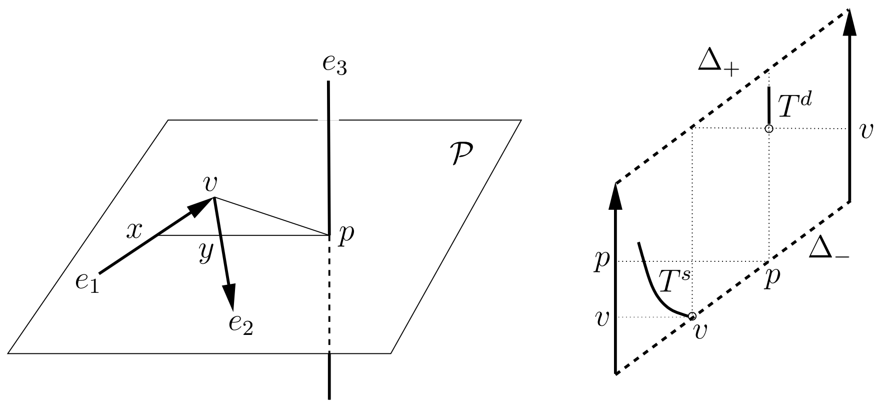

Figure 2.13: The left picture shows an interval of adjacent trisecants ending on degenerate trisecant vvp or pvv. The right picture shows the corresponding intervals of trisecants in S. The intervals of T^s and T^d correspond to trisecants with linear ordering $e_1e_2e_3$ and $e_3e_2e_1$ respectively.

the trisecant ($x \in e_1$) moves with the parameterization of K and the second ($y \in e_2$) against. Thus x and y are both close together on K and y is just after x in the orientation of the knot. Thus $y > x$ in K^2 and approaches the diagonal from above. Hence in S, this interval of trisecants has a negative slope and ends at $vv \in \Delta_-$. In this situation $\pi_{12}(\partial\mathcal{T}) \subset \Delta_-$. Now with the orientation of K remaining the same, trisecants whose points lie on $e_3e_2e_1$ in that order are trisecants of different ordering T^d. The first point is fixed and the second moves against the parameterization. Hence this interval of trisecants is a vertical line, decreasing and ending at the point $pv \in S$ and $\pi_{12}(\partial\mathcal{T}) \subset S$. Figure 2.13 (right) shows the two corresponding intervals of trisecants in S.

We repeat the discussion but with the orientation of K reversed moving from e_2 to e_1. Reversing the orientation of K interchanges the sets T^s and T^d. Hence trisecants whose points lie on edges $e_1e_2e_3$ in that order are now in T^d. The first point of the trisecant ($x \in e_1$) moves against the parameterization of K, the second ($y \in e_2$) with it. Again x and y are close together on K and y is just before x in the orientation of K. Thus $y < x$ in K^2 and approaches the diagonal from below. Hence the interval of trisecants has negative slope and ends on $vv \in \Delta_+$ and in this situation $\pi_{12}(\partial\mathcal{T}) \subset \Delta_+$. Now keep the orientation of K from e_2 to e_1. The trisecants whose points lie on edges $e_3e_2e_1$ in that order are in T^s. The first point of the trisecant is fixed, the second moves with the parameterization of K. Hence the interval of trisecants is a vertical line increasing and ending at $pv \in S$ and $\pi_{12}(\partial\mathcal{T}) \subset S$.

We have completely described the behavior of $\pi_{12}(\partial\mathcal{T})$ in $\overline{S}$. Let $\partial T := \pi_{12}(\partial\mathcal{T})$, then ∂T lies in the interior of S as the limit point of a vertical interval of trisecants whose first point is fixed. Vertical intervals of trisecants cannot end on Δ_+ or Δ_-. Similarly, negatively sloped intervals of trisecants must end on Δ_+ or Δ_- not in the interior of S. Here ∂T lies on Δ_+ or Δ_- as the limit point of a negatively sloped interval of trisecants whose third point is fixed. Hence $\overline{T} := \pi_{12}(\overline{\mathcal{T}})$ has been defined on $\overline{S}$.

Definition 2.3.5. Recall $\overline{T} = \pi_{12}(\overline{\mathcal{T}})$ and similarly we define $\overline{T}^s = \pi_{12}(\overline{\mathcal{T}}^s)$ and $\overline{T}^d = \pi_{12}(\overline{\mathcal{T}}^d)$. Also $\partial T := \pi_{12}(\partial\mathcal{T})$ and similarly we define $\partial T^s = \pi_{12}(\partial\mathcal{T}^s)$ and $\partial T^d = \pi_{12}(\partial\mathcal{T}^d)$.

Now $\overline{S}$ is a compact metric space, and $\overline{T}^s$ and $\overline{T}^d$ are closed, (and hence) compact subsets of $\overline{S}$. We apply Lemma 2.3.6 (below) to $\overline{T}^s$ and $\overline{T}^d$ in $\overline{S}$. Thus $d(T^s, T^d) = d(\overline{T}^s, \overline{T}^d)$ and $d(\overline{T}^s, \overline{T}^d) = 0$ if and only if $\overline{T}^s \cap \overline{T}^d \neq \emptyset$. In Lemma 2.3.16 we will show that if $\overline{T}^s \cap \overline{T}^d \neq \emptyset$, then $T^s \cap T^d \neq \emptyset$. Hence by Lemma 2.1.6 it will be sufficient to show that $\overline{T}^s \cap \overline{T}^d \neq \emptyset$ in $\overline{S}$ in order to show that an alternating quadrisecant exists.

Lemma 2.3.6. *Suppose X is a compact metric space and $U, V \subset X$. Then $d(U, V) = d(\overline{U}, \overline{V})$ and this is zero if and only if $\overline{U} \cap \overline{V} \neq \emptyset$.*

Proof. See any standard topology text such as [Dug]. □

Lemma 2.3.7. *Let K be a generic polygonal knotted curve. The projection π_{12} is a piecewise smooth immersion of $\mathcal{T}$ into S.*

Proof. The map $\pi_{12}(\mathcal{T})$ is just an orthogonal projection. Recall that $\mathcal{T}$ is an embedded 1-manifold in K^3. From Lemma 2.3.1 and Lemma 2.3.2 we know that $\mathcal{T} \cap (e_i \times e_j \times e_k)$ is both smooth and monotonic in at least two variables of $e_i \times e_j \times e_k$. Thus the tangent vector is never vertical. The projection π_{12} of $\mathcal{T} \cap (e_i \times e_j \times e_k)$ in S is a smooth immersion. Thus π_{12} is a piecewise smooth immersion of $\mathcal{T}$ in S. □

We referred to $\mathcal{T} \cap (e_i \times e_j \times e_k)$ as an interval of trisecants. We will also refer to an interval of trisecants or an interval of T in S when we consider the image of $\mathcal{T} \cap (e_i \times e_j \times e_k)$ under π_{12}.

The following two lemmas show that in S, T can intersect itself only in double-points. This means that quadrisecants (if they exist) are isolated.

Lemma 2.3.8. *Let K be a generic polygonal knotted curve. In S, the intersection of two intervals of T (which do not share a common vertex) is either empty, one point, or two distinct points.*

Proof. In S, one interval of trisecants will contain trisecants whose points lie on edges e_1, e_2, e_3, the points of the other interval lie on edges e_1', e_2', e_3'. These intervals have a point in common in S if and only if two trisecants have the same first and second points. That is, only if they have the same first and second edges: so $e_1 = e_1'$ and $e_2 = e_2'$. However $e_3 \neq e_3'$, or else there would not be two intervals of T intersecting in S. Thus two intersecting intervals of T in S have the same first two edges and are distinguished by their third edges.

If two intervals of trisecants do not have the same first two edges, then their projections will not intersect on S. However, having the same first two edges may or may not guarantee an intersection. There are three cases to consider. In all cases the first and second edges are the same and the third edges differ. Let the first two edges be denoted e_1 and e_2 and let the differing third edges be denoted by e_3 and e_3'.

Case 1: Both intervals of trisecants are adjacent trisecants. There is only one possible case, the first two edges are adjacent. (If e_1 and e_3 are adjacent and e_1 and e_3' are adjacent, then a quadrisecant whose points lie on $e_1e_2e_3e_3'$ includes a trisecant whose points lie on three adjacent edges $e_1e_3e_3'$, contradicting non-degeneracy.) Thus both intervals of trisecants lie in the plane $\mathcal{P}$ spanned by e_1 and e_2. Let e_3 intersect $\mathcal{P}$ in the point p and e_3' intersect $\mathcal{P}$ in the point p'. Figure 2.14 illustrates this. The line through pp' is the only possible common trisecant line. Hence in S these intervals of trisecants intersect in at most one point.

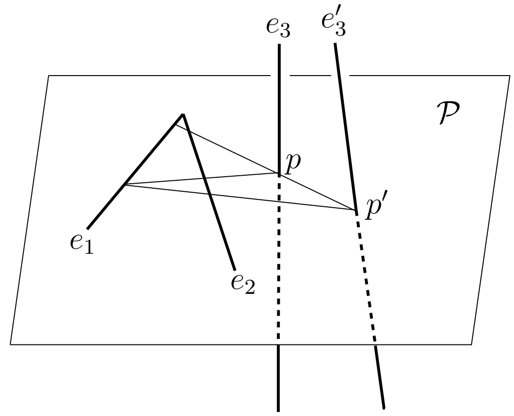

Figure 2.14: Trisecants with same two adjacent first edges can have at most one line in common.

Case 2: One interval of trisecants consists of adjacent trisecants and the other of skew trisecants. Let the points of the skew trisecant lie on edges e_1, e_2, e_3. The pairwise skew lines determined by e_i generate a doubly-ruled surface. Let the points of the adjacent trisecants lie on edges e_1, e_2 and e_3'. As e_1 and e_2 are pairwise skew, either e_1 and e_3' are adjacent, or e_2 and e_3' are adjacent. In either case, the plane spanned by the two adjacent edges intersects the doubly-ruled surface in some quadratic curve. However, as one of the edges lies in the doubly-ruled surface, the curve degenerates to a pair of straight lines, one of which includes edge e_1 (or e_2). The other straight line is the only possible common trisecant line and only if it is both a trisecant line on the doubly-ruled surface and the plane. Thus the intervals of trisecants intersect in at most one point.

Case 3: Both intervals of trisecants consist of skew trisecants. Let the points of one interval of skew trisecants lie on edges e_1, e_2, e_3. Let H be the doubly-ruled surface generated by the lines containing the polygonal edges e_i. Suppose e_3' intersects H, it does so transversely in one or two points. The trisecant lines(s) through these point(s) will only correspond to quadrisecant line(s) if it is a trisecant through the edges e_1, e_2, e_3, *and* through the edges e_1, e_2, e_3'. In fact this is all that can happen. The edge e_3' can never be tangent to nor contained in H. If it were, then either its vertices lie in H or it is tangent to H, which contradicts genericity condition (G1) on page 28. Thus in all three cases the intersection of two intervals of T in S is either empty, one point or two distinct points. □

Remark 2.3.9. It is worth discussing intervals of trisecants which do have a vertex in common. Again, suppose the points of one interval of trisecants lie on edges e_1, e_2, e_3 and suppose the points of the other interval of trisecants lie on edges e'_1, e'_2, e'_3. Assume that these intervals of trisecants have a vertex in common. From Proposition 2.3.4 and Lemmas 2.3.1 and 2.3.2 this common vertex is a vertex trisecant with vertex v, say. (It is not a quadrisecant!) From the discussion in the proof of Proposition 2.3.4, v belongs to two of the edges and the remaining edges are identical. There are three cases:

Case 1: $v \in e_1, e'_1$ and $e_2 = e'_2$, $e_3 = e'_3$.

Case 2: $v \in e_2, e'_2$ and $e_1 = e'_1$, $e_3 = e'_3$.

Case 3: $v \in e_3, e'_3$ and $e_1 = e'_1$, $e_2 = e'_2$.

In Case 1 (respectively Case 2) the first (respectively second) points of the trisecants lie on different edges. Thus the only possible common point in S is the projection of the vertex trisecant.

In Case 3, the first two edges are the same; thus there is some possibility that the intervals of trisecants may intersect in S. The arguments proceed just as in Lemma 2.3.8 so we only outline them here. Lemma 2.3.8 Case 1 gives the same result — there is at most one point of intersection (apart from the common vertex trisecant). Lemma 2.3.8 Case 2 turns out to be a situation where e_3 and e'_3 cannot have a common vertex. Suppose they did, then edges $e_1 e_3 e'_3$ (or $e_2 e_3 e'_3$) are adjacent; thus the quadrisecant whose points lie on edges $e_1 e_2 e_3 e'_3$ includes a trisecant whose points lie on three consecutive edges, contradicting non-degeneracy. Lemma 2.3.8 Case 3 may be repeated, except that as e'_3 shares a common vertex with e_3, it intersects the doubly-ruled surface H in at most one point (rather than two). Thus intervals of trisecants which share a common vertex have at most one other common point when projected under π_{12} to S.

Lemma 2.3.10. *Let K be a generic polygonal knotted curve. No more than two points of $\mathcal{T}$ can have the same image under π_{12}.*

Proof. Trisecants have the same image under π_{12} if their first two points lie on the same first two edges and the third points lie on different edges. If there were three (or more) points of $\mathcal{T}$ with the same image under π_{12}, this would correspond to a quintisecant (or higher order secant) in contradiction to genericity condition (G2) on page 28. (See also Figure 2.16A.) □

Lemma 2.3.11. *Let K be a generic polygonal knotted curve. If t is a vertex trisecant, then no other trisecant $t' \in \mathcal{T}$ has the same image under π_{12}.*

Proof. Let $t, t' \in \mathcal{T}$, then $\pi_{12}(t) = \pi_{12}(t')$ implies that t and t' are included in a quadrisecant. As t is also a vertex trisecant, then the quadrisecant will have one (or two) vertices of K.

Case 1: Suppose t' is an adjacent trisecant as in Lemma 2.3.1. Then two of t's points lie in two adjacent edges and t' lies in the plane $\mathcal{P}$ spanned by these edges. Suppose $\pi_{12}(t) = \pi_{12}(t')$, then t also lies in $\mathcal{P}$, in particular, the vertex of t lies in $\mathcal{P}$. Suppose this vertex is a vertex of one of the two adjacent edges. If it is the common vertex of the adjacent edges, then the quadrisecant (including t and t') lies in the osculating plane of the vertex, a contradiction to genericity condition (G3) on page 28. If the vertex is one of the other vertices, then the quadrisecant contradicts genericity condition (G4) on page 28. The only possibility remaining is that the vertex belongs to one of the other edges intersecting $\mathcal{P}$. In this case, more than four vertices are coplanar, contradicting non-degeneracy.

Case 2: Suppose t' is a skew trisecant as in Lemma 2.3.2. There are three pairwise skew edges which contain the points of t' and generate a doubly-ruled surface H. Suppose $\pi_{12}(t) = \pi_{12}(t')$, then t also lies in H; in particular, the vertex of t lies in H. But, from genericity condition (G1) on page 28 we know that the only vertices which lie on H are those belonging to the three generating edges. Suppose that the vertex of t is one of these vertices; then one of the other two points of t lies on a fourth edge of K. Thus t's vertex either lies on yet another doubly-ruled surface or lies in a plane spanned by two adjacent edges. Both of these cases have already been eliminated. □

Corollary 2.3.12. *A generic polygonal knot has no vertex quadrisecants.*

Proof. For $abcd$ to be a vertex quadrisecant, exactly one of a, b, c or d must be a vertex of K. Thus either abc and/or abd is a vertex trisecant and $\pi_{12}(abc) = \pi_{12}(abd)$, in contradiction to Lemma 2.3.11. □

We conjecture that the set of trisecants intersects itself transversely in S. This conjecture is very close to being proved. We state the conjecture below and give a complete proof for Case 1 and Case 2. The proof of Case 3 is outlined with the final calculation omitted.

Conjecture 2.3.13. *Let K be a generic polygonal knotted curve. In S, two intervals of T either join at a vertex trisecant, intersect transversely in their interiors or both.*

Proof. Lemma 2.3.8, Remark 2.3.9 and Lemma 2.3.10 showed that in S, two intervals of T either join via a vertex trisecant and intersect in at most one point, or two intervals of T intersect in at most two distinct points. These intersection points represent quadrisecants. Moreover, Lemma 2.3.11 showed that such intersection points occur in the interior of the intervals of T — there are no vertex quadrisecants. What remains to be shown is the transversality of interior intersections.

Let $t, t' \in \mathcal{T}$ be trisecants from each interval of T such that $\pi_{12}(t) = \pi_{12}(t')$. To show that intervals of trisecant cross transversely at $\pi_{12}(t) = \pi_{12}(t')$, we need to show that the tangent directions (of the intervals of T) at this point are different. As in Lemma 2.3.8, the points of trisecants t and t' lie on the same first two edges e_1 and e_2 and on differing third edges e_3 and e_3' respectively. There are three cases to consider. Firstly, both intervals of trisecants consist of adjacent trisecants. Secondly, one interval of trisecants consists of skew trisecants and the other of adjacent trisecants. Thirdly both intervals of trisecants consist of skew trisecants.

Case 1: Both intervals of trisecants consist of adjacent trisecants. Throughout the following argument it will be helpful to refer to Figure 2.14 on page 45. As in Lemma 2.3.8, edges e_1 and e_2 have a common vertex and edges e_3 and e_3' are distinct. Let the plane spanned by e_1 and e_2 be denoted $\mathcal{P}$ and let p be the point where e_3 intersects $\mathcal{P}$ and p' be the point where e_3' intersects $\mathcal{P}$. One interval of trisecants intersects the interior of edges e_1, e_2, e_3 and the other edges e_1, e_2, e_3'. From the Appendix we know that for an interval of adjacent trisecants, the ratio of arclength of e_1 and e_2 depends on trig factors and the distance from the vertex to the point p (or p'). Now both the trig factors and the distances are different for p and p'. In the Appendix we show this means the tangent directions of the intervals of trisecants are different at $\pi_{12}(t)$ and $\pi_{12}(t')$. Hence in S, the two intervals of T are transverse to each other.

Case 2: One interval of trisecants consists of skew trisecants and the other of adjacent trisecants. Suppose edges e_1, e_2 and e_3 are pairwise skew and generate the doubly-ruled surface H. The points of the adjacent trisecants lie on edges e_1, e_2, e_3'. Thus either e_1 and e_3' are adjacent edges or e_2 and e_3' are adjacent edges. In S, the interval of trisecants whose points lie on e_1, e_2 and e_3' is a horizontal interval when e_1 and e_3' are adjacent and is a vertical interval when e_2 and e_3' are adjacent. In S, the interval of trisecants whose points lie on e_1, e_2 and e_3 is smooth and monotonic in *both* e_1 and e_2. Thus it must be transverse to the other interval (either horizontal or vertical).

Case 3: Both interval of trisecants consist of skew trisecants. Note that this proof is in outline and the necessary calculations omitted. Suppose edges e_1, e_2 and e_3 generate the doubly-ruled surface H and edges e_1, e_2 and e_3' generate the doubly-ruled surface H'. As $\pi_{12}(t) = \pi_{12}(t')$ these surfaces intersect along a quadrisecant line. From genericity condition (G1) edge e_3' intersects H transversely. We wish to show this implies that H and H' intersect transversely along the quadrisecant line and that the intervals of trisecants intersect transversely in S.

Consider the two doubly-ruled surfaces H and H'. They both have e_1 and e_2 in one of their rulings. We have assumed that e_3' is not contained in H; thus $H \neq H'$. Recall that each point of a doubly-ruled surface lies on one line from each ruling. These lines span the tangent plane to that point on the surface. Now along e_1 and e_2 the other rulings of H and H' are for the most part different. From Lemma 2.3.8 we know there are at most two quadrisecant lines through e_1 and e_2. Thus there are at most two lines in common of the other rulings of H and H' through e_1 and e_2. Apart from these quadrisecant lines, the tangent planes are different for H and H' along e_1 and e_2 . To see this, take a point q on e_1. The lines l and l' through q, from the other rulings of H and H' respectively, intersect e_2 in different points. Thus l and l' span a plane. As e_1 and e_2 are skew, e_1 does not lie in this plane. Thus the planes spanned by e_1 and l and e_1 and l' are different. A similar argument holds for e_2. Hence the tangent planes to H and H' along e_1 and e_2 are different — apart from those points from the quadrisecant lines — and H and H' intersect transversely along e_1 and e_2.

Let E_3' be the line determined by e_3'. Then E_3' either does not intersect H, is tangent to H at one point, or intersects H transversely in two distinct points (one possibly at infinity). We must be in last case, as we assumed e_3' is not tangent to H by genericity condition (G1). If E_3' is transverse to H, then the tangent planes of H and H' differ (at both points of intersection). Note that if H and H' are not transverse along the quadrisecant line, then they share common tangent planes along the quadrisecant line. Hence e_3' is tangent to H (and also e_3 tangent to H'). This contradicts genericity condition (G1) on page 28. (A similar argument could be made at each point of the quadrisecant line, the lines of the other ruling of H must not be tangent to H' along the quadrisecant line and vice versa.)

Assume for both H and H' that the arclength changes with unit speed along e_1. From the Appendix we know that along the interval of trisecants the corresponding points of e_1 and e_2 change smoothly and monotonically. We are interested in how arclength of e_2 changes as e_1 changes. The change in the tangent planes capture this. We see that before and after the common tangent plane at the quadrisecant line the tangent planes are different. Thus the change is different. We have omitted the final calculation which shows that this implies that the tangent vectors to the intervals of trisecants must be different. Completing this calculation will show that intervals of trisecants must intersect transversely in S. □

Let $\pi_{ij}(K^3) \to K^2$ be projection onto the ith and jth coordinates (where $i < j$ and $i, j = 1, 2, 3$). We have extensively studied the structure of $T = \pi_{12}(\mathcal{T})$ for generic polygonal knots. We have shown that it is the image of a piecewise smooth immersion of $\mathcal{T}$ and T intersects itself in double points. We can also say the same thing about $\pi_{13}(\mathcal{T})$ and $\pi_{23}(\mathcal{T})$. Lemma 2.3.7 carries over immediately. Similarly Lemma 2.3.8, Remark 2.3.9, Lemma 2.3.10, Lemma 2.3.11 and Conjecture 2.3.13 can be proved for $\pi_{13}(\mathcal{T})$ and $\pi_{23}(\mathcal{T})$ with only minor alterations. Thus the following proposition holds.

Proposition 2.3.14. *The projection π_{ij} ($i < j$ and $i, j = 1, 2, 3$) is a piecewise smooth immersion of $\mathcal{T}$ into S and $T = \pi_{ij}(\mathcal{T})$ intersects itself (transversely) in double points.*

For the most part we are only interested in $T = \pi_{12}(\mathcal{T})$ in S. Recall that $\pi_{12}(\mathcal{T})$ allows us to capture the orderings of coincident trisecants which in turn enables us to find alternating quadrisecants. In Lemma 2.3.16 we will show that if $\overline{T}^s \cap \overline{T}^d \neq \emptyset$ on $\overline{S}$, then $T^s \cap T^d \neq \emptyset$ in S. Thus it is important to understand exactly where $\overline{T}^s \cap \overline{T}^d \neq \emptyset$ but T^s and T^d do not intersect. Recall $\overline{T} = \pi_{12}(\overline{\mathcal{T}}) = \pi_{12}(\mathcal{T} \cup \partial\mathcal{T})$. After Proposition 2.3.4 we showed that $\pi_{12}(\partial\mathcal{T})$ lies on Δ_+ or Δ_- as the limit point of an interval of trisecants whose first two points both end on a vertex of K and whose third point is fixed. If the second and third points of the trisecants both end on a vertex of K and the first is fixed, then $\pi_{12}(\partial\mathcal{T})$ lies in the interior of S.

The following lemma shows that $\overline{T}^s$ stays away from $\tilde{\Delta}_+$ and $\overline{T}^d$ stays away from $\tilde{\Delta}_-$. Hence $\overline{T}^s$ and $\overline{T}^d$ cannot intersect on Δ_+ or Δ_-, but only in the interior of S.

Lemma 2.3.15. *Let K be a generic polygonal curve. In S, $d(\overline{T}^d, \tilde{\Delta}_-) \geq h$ and $d(\overline{T}^s, \tilde{\Delta}_+) \geq h$, where h is the minimum edge length of K.*

Proof. As one approaches $\tilde{\Delta}_-$ it is the same as approaching the diagonal in K^2 from above. (Thus for secant xy, points x and y are close together and $y > x$ so y is just *after* x in the order of the knot.) Now consider what it means for secant xy to also be in T^d. The order along the knot is xzy (where z is the third point of the trisecant). Thus for $xy \in T^d$, x and y are separated by at least one edge as z must come in between them and by definition,

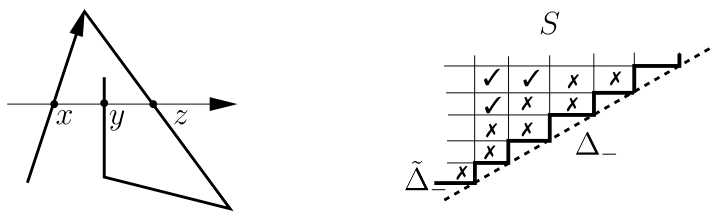

Figure 2.15: The first two points of trisecants of different order must be at least two edge lengths apart. This information is shown on part of the set of secants S on the right. The ticked boxes indicate where trisecants of different order may be.

distinct points of trisecants lie on distinct edges. In fact x and y must be separated by at least two edges. If they were just separated by one edge, then the points of the trisecant xyz lie on three consecutive edges contradicting non-degeneracy. Figure 2.15 (left) illustrates this and Figure 2.15 (right) shows this information recorded on S. Trisecants of different order may be in boxes marked with a tick, not a cross. We see that trisecants of different order are at least one edge away from $\tilde{\Delta}_-$ (indicated by a bold line). Hence $d(T^d, \tilde{\Delta}_-) \geq h$, where h is the minimum edge length of K. In an analogous way $d(T^s, \tilde{\Delta}_+) \geq h$. $\square$

Lemma 2.3.16. *Let K be a generic polygonal curve. If $\overline{T}^s \cap \overline{T}^d \neq \emptyset$ in $\overline{S}$, then $T^s \cap T^d \neq \emptyset$ in S.*

Proof. Lemma 2.3.15 shows that $\overline{T}^s$ and $\overline{T}^d$ can only intersect in the interior of S. Suppose $t \in \overline{\mathcal{T}}^s$ and $t' \in \overline{\mathcal{T}}^d$ with $\pi_{12}(t) = \pi_{12}(t') \in S$. We want to show $t \in \mathcal{T}^s$ and $t' \in \mathcal{T}^d$. Assume, by way of contradiction that $t \in \partial\mathcal{T}^s$ and consider two cases: $t' \in \mathcal{T}^d$ and $t' \in \partial\mathcal{T}^d$.

Case 1: Assume that $t \in \partial\mathcal{T}^s$ and $t' \in \mathcal{T}^d$ and $\pi_{12}(t) = \pi_{12}(t') \in S$. Figure 2.16B illustrates this case.

Let $ab = \pi_{12}(t) = \pi_{12}(t')$. As $t \in \partial\mathcal{T}^s$, there is a vertical interval of trisecants which increases to and ends at secant ab in S. Thus at ab there is a vertex of the knot at b. Let a belong to edge e_a and b be the vertex belonging to edges e_{b_1} and e_{b_2}. The vertical open interval of trisecants corresponds to trisecants with first point a and second and third points on edges e_{b_1} and e_{b_2} which ends at degenerate trisecant $abb \in K^3$ or secant $ab \in S$. But $ab = \pi_{12}(t')$, where $t' \in \mathcal{T}^d$. Thus ab is also the first two points of a trisecant of

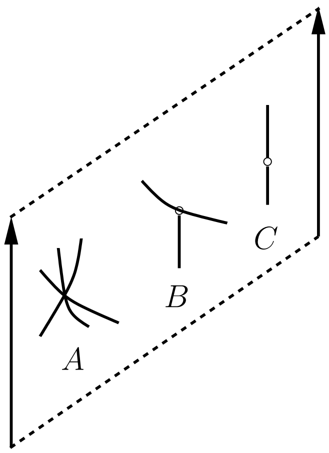

Figure 2.16: Intersections of T in S forbidden for a generic polygonal knot.

different order abc, where c lies on edge e_c. Secant ab, and hence trisecant abc lies in the plane spanned by e_{b_1} and e_{b_2}. Thus trisecant abc lies in the osculating plane of vertex b, contradicting genericity condition (G3) on page 28.

Case 2: Assume that $t \in \partial\mathcal{T}^s$ and $t' \in \partial\mathcal{T}^d$ and $\pi_{12}(t) = \pi_{12}(t') \in S$. Figure 2.16C illustrates this case.

Let $ab = \pi_{12}(t) = \pi_{12}(t')$. As $t \in \partial\mathcal{T}^s$ and $t' \in \partial\mathcal{T}^d$, ab belongs to two vertical intervals of trisecants which both end at secant ab in S. The trisecants of same and different order both have a as a starting point and both have points which lie on edges which have b as a common vertex. But there are only two edges with b as a common vertex. This implies t and t' have the same first point and both have second and third points on the same two edges. This implies that the two interval of trisecants are in fact the same (and hence have the same order). This contradicts the fact t and t' are trisecants of same and different order respectively. □

In Lemma 2.1.6 we showed that an alternating quadrisecant exists when T^s and T^d have a common point in S. In Section 2.2 and Section 2.3 we examined the structure of the set of trisecants for generic polygonal knots. Through Lemma 2.3.16 we see that it is sufficient to prove $\overline{T}^s$ and $\overline{T}^d$ have a common point in $\overline{S}$ in order to show that an alternating quadrisecant exists. In fact, we will prove a stronger result than this in Section 3.2.

Chapter 3

Essential Alternating Quadrisecants

3.1 Essential secants and trisecants

Eventually we want to prove that all nontrivial tame knots have an alternating quadrisecant. To do this we will prove that nontrivial generic polygonal knots have an *essential* alternating quadrisecant and then use a limit argument. We will define essential shortly — this is the topological notion required so the limit argument works.

In Section 2.2 we showed that the set of generic polygonal knots is open and dense in the set of polygonal knots. The set of polygonal knots is dense in the set of all tame knots, thus the set of generic polygonal knots is dense in the set of all tame knots. For any knotted curve K, there is a sequence of generic polygonal knots $\{K_i\}_{i=1}^{\infty}$ which converge to K. In the proof of Theorem 4.1.3 we show how this sequence of K_i may be chosen so that K_i converges in C^0 and so that each K_i is ambient isotopic to K.

Thus we have $K_i \to K$. Theorem 3.2.4 shows that each generic polygonal knot K_i has an essential alternating quadrisecant $a_i b_i c_i d_i$. As each K_i is contained in some ball in $\mathbb{R}^3$, eventually the entire sequence $\{K_i\}_{i=1}^{\infty}$ lies in a common compact subset of $\mathbb{R}^3$. By taking a subsequence if necessary, there is a sequence of alternating quadrisecants which converge to some quadrisecant $abcd \in K^4$. We may also assume that the order of $a_i b_i c_i d_i$ along the knot is $a_i c_i b_i d_i$ (see Figure 3.3).

Lemma 3.1.1. *Given the sequence of generic polygonal knots with converging alternating quadrisecants as described above. If b does not lie on the same edge as c, then no two limit points lie on the same edge of K and so $abcd$ is a quadrisecant.*

Proof. Given two points $p, q \in K$, we denote $p \sim q$ to mean p and q lie on the same common straight subarc of K. Quadrisecant points lie on the same straight subarc of K in the limit if and only if they are next to each other in the ordering of *both* the quadrisecant line and the knot. Thus $b \nsim c$ implies $a \nsim b$ as the order along the knot is $a_i c_i b_i$. Similarly $b \nsim c$ implies $c \nsim d$ as the knot order is $c_i b_i d_i$ and $b \nsim c$ implies $a \nsim d$ as the knot order is $a_i c_i b_i d_i$. □

In order to show the limit $abcd$ is a quadrisecant, we only have to worry about whether the middle two intersection points b and c merge together. E. Pannwitz [Pann] proved that almost all knots have a quadrisecant and G. Kuperberg [Kup] proved that all (nontrivial tame) knots have quadrisecants. He did this by using the limit argument outlined above. In order to show the limit was a quadrisecant, he introduced the notion of essential secants and quadrisecants (which he called "topologically nontrivial"). As we have encountered the same problem, we also use the notion of essential, but redefine and extend it to suit our purposes. We start by defining when an arc of a knot is essential, capturing part of the knottedness of K. Generically, a knot K together with a secant segment $S = \overline{pq}$ forms a knotted Θ-graph in space. To adapt G. Kuperberg's definition, we consider such knotted Θ-graphs.

Definition 3.1.2. Suppose α, β and γ are three disjoint simple arcs from p to q, forming a knotted Θ-graph. Then we say that the ordered pair (α, β) is **inessential** if there is a disk D bounded by the knot $\alpha \cup \beta$ having no interior intersections with the knot $\alpha \cup \gamma$. (We allow self-intersections of D, and interior intersections with β, as will be necessary if $\alpha \cup \beta$ is knotted.)

An equivalent definition is illustrated in Figure 3.1: Let $X := \mathbb{R}^3 \smallsetminus (\alpha \cup \gamma)$, and let δ be a parallel curve to $\alpha \cup \beta$ in X. Here by **parallel** we mean that $\alpha \cup \beta$ and δ cobound an annulus embedded in X. We choose δ so that it is homologically trivial in X (that is, so

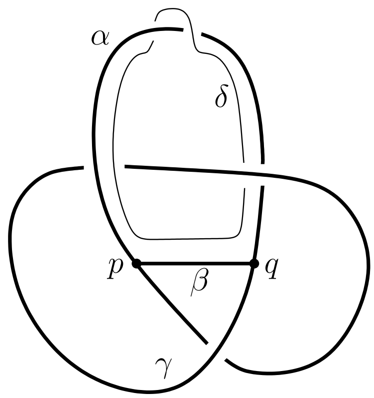

Figure 3.1: In the knotted Θ-graph $\alpha \cup \beta \cup \gamma$, the ordered pair (α, β) is essential. To see this, we find the parallel $\delta = h(\alpha, \beta)$ to $\alpha \cup \beta$ which has linking number zero with $\alpha \cup \gamma$ and note that is is homotopically nontrivial in the knot complement $\mathbb{R}^3 \smallsetminus (\alpha \cup \gamma)$. In this illustration, β is the straight segment $\overline{pq}$, so we may equally say that the arc α of the knot $\alpha \cup \gamma$ is essential.

that δ has linking number zero with $\alpha \cup \gamma$). Let $h(\alpha, \beta) \in \pi_1(X)$ denote the (free) homotopy class of δ. Then (α, β) is **inessential** if $h(\alpha, \beta)$ is trivial. We say that (α, β) is **essential** if it is not inessential.

Now let λ be a meridian loop (linking $\alpha \cup \gamma$) in the knot complement X. If the commutator $[\lambda, h(\alpha, \beta)]$ is nontrivial then we say (α, β) is **strongly essential**.

This notion is clearly a topological invariant of the (ambient isotopy) class of the knotted Θ-graph. We apply this definition to arcs of a knot K below. But first, some useful notation. Let $p, q \in K$. The arc from p to q following the orientation of the knot is denoted γ_{pq}. The arc from q to p, γ_{qp} is similarly defined. The secant segment from p to q is denoted $\overline{pq}$.

Definition 3.1.3. If K is a knot and $p, q \in K$, let $S = \overline{pq}$. Assuming S has no interior intersections with K, we say that γ_{pq} is **(strongly) essential** for K if (γ_{pq}, S) is essential in the knotted Θ graph $K \cup S$. If S does intersect K, then we say γ_{pq} is **(strongly) essential** if for any $\epsilon > 0$ there exists some ϵ-perturbation of S (in the C^1 sense, with endpoints fixed) to a curve S' such that $K \cup S'$ forms an embedded Θ in which (γ_{pq}, S') is (strongly) essential

In [CKKS] it was shown that if K is an unknot, then any arc γ_{pq} is inessential. This follows immediately, because the homology and homotopy groups of $X := \mathbb{R}^3 \smallsetminus K$ are equal

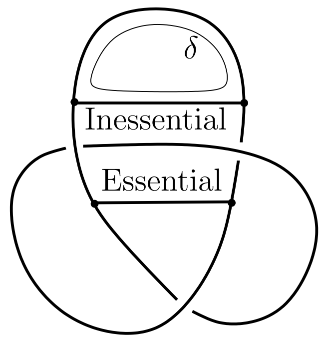

Figure 3.2: Examples of essential and inessential secants for the trefoil knot. The parallel curve δ is shown for the inessential secant.

for an unknot, so any curve δ having linking number zero with K is homotopically trivial in X.

We can use Dehn's lemma (Lemma 1.2.20) to prove a converse statement. Recall Dehn's lemma provides a practical criterion of triviality for a knot in $\mathbb{R}^3$. To show a knot K is trivial, it is sufficient to show it has a spanning disk with no self-intersections near the boundary. (The disk may have self-intersections in its interior.)

Theorem 3.1.4. *If $p, q \in K$ and both γ_{pq} and γ_{qp} are inessential, then K is unknotted.*

Proof. Let S be the secant segment $\overline{pq}$ perturbed if necessary to avoid interior intersections with K. We know that $\gamma_{pq} \cup S$ and $\gamma_{qp} \cup S$ bound disks D' and D'' whose interiors are disjoint from K. Glue these two disks together along S to form a disk D spanning K. This disk may have self intersections, but these occur away from the boundary K. By Dehn's lemma, we can replace D by an embedded disk, so K is unknotted. □

Definition 3.1.5. A secant pq of K is **essential** if both sub-arcs γ_{pq} and γ_{qp} are essential. Otherwise it is **inessential**. Let ES be the **set of essential secants in** S. Figure 3.2 illustrates an inessential and an essential secant of a trefoil knot.

To call a quadrisecant *abcd* essential, we could follow G. Kuperberg [Kup] and require that secants *ab*, *bc* and *cd* all be essential. Recall that we need the notion of essential so that *abcd* will be a quadrisecant in the limit. Lemma 3.1.1 showed that for alternating

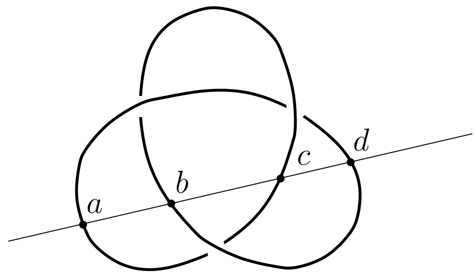

Figure 3.3: Essential alternating quadrisecant *abcd*.

quadrisecants it suffices for secant bc to be essential. (This is because b and c lie next to each other in the ordering of both the quadrisecant line and the knot.)

Definition 3.1.6. An **essential trisecant** abc is a trisecant which is essential in the second segment bc. An **essential alternating quadrisecant** $abcd$ is an alternating quadrisecant which is essential in the second segment bc. (See Figure 3.3.) Trisecants and alternating quadrisecants that are not essential are called **inessential**.

We show in the proof of Main Theorem that when taking the limit of essential secants, the points of the limit secant do not combine to form one point, nor lie on a common edge of K. Thus when taking the limit of essential alternating quadrisecants, we may be assured of an alternating quadrisecant. We want to prove that any nontrivial generic polygonal knot has an *essential* alternating quadrisecant. Thus we need to understand the relationship between essential trisecants and essential quadrisecants.

Definition 3.1.7. Let $\mathcal{ET}$ be the set of essential trisecants in K^3, really, $\mathcal{ET} = \pi_{23}^{-1}(ES) \cap \mathcal{T}$. Define $\mathcal{ET}^s$ and $\mathcal{ET}^d$ to be the sets of essential trisecants of same and different orderings in K^3. Let $ET = \pi_{12}(\mathcal{ET})$ be the projection of the set of essential trisecants to the set of secants S and similarly define $ET^s := \pi_{12}(\mathcal{ET}^s)$ and $ET^d := \pi_{12}(\mathcal{ET}^d)$.

Lemma 3.1.8. *Let $ab \in ET^s \cap ET^d$ in S. This means that there exists c and d such that $abc \in \mathcal{ET}^d$ and $abd \in \mathcal{ET}^s$. Then either $abcd$ or $abdc$ is an essential alternating quadrisecant.*

Proof. That $abcd$ or $abdc$ is an alternating quadrisecant follows from Lemma 2.1.6. To show it is essential, the midsegment bc, respectively bd, must be essential. The order along the quadrisecant line is either $abcd$ or $abdc$. If it is $abcd$ then bc is essential as $abc \in \mathcal{ET}^d$. If it is $abdc$, then bd is essential as $abd \in \mathcal{ET}^s$. □

Thus in order to show that a generic polygonal knot has at least one essential alternating quadrisecant, it is sufficient to prove that ET^s and ET^d have common points in S. Note that it is **not** enough to show that $ET^s \cap T^d \neq \emptyset$ (or vice versa). Let $ab \in ET^s \cup T^d$ and let $abc \in \mathcal{T}^d$ and $abd \in \mathcal{ET}^s$. Then alternating quadrisecant $abcd$ is not necessarily essential, as bc is not necessarily essential.

In the following lemmas we describe the relationship between the set of essential trisecants and the set of trisecants. We use this relationship to completely understand the structure of ET in S.

Lemma 3.1.9. *Let K be a generic polygonal knot. In $S \setminus \pi_{13}(\mathcal{T})$, each connected component consists of either inessential or essential secants. Moreover, in S, the set of essential secants is closed.*

Proof. Recall that $\pi_{13} : K^3 \to K^2$ is projection onto the first and third coordinates. From Proposition 2.3.14 we know that π_{13} piecewise smooth immersion of $\mathcal{T}$ in S and $\pi_{13}(\mathcal{T})$ intersects itself in double points. (Thus $S \setminus \pi_{13}(\mathcal{T})$ consists of connected components.)

We wish to show that in S, secants change from inessential to essential *only* when there is a trisecant $t \in \pi_{13}(\mathcal{T})$. Let ab be an inessential (essential) secant. By perturbing $a, b \in K$ we can get to all nearby secants in S. As long as the chord $\overline{ab}$ never touches K during the perturbation then the topological type of the knotted Θ-graph has not changed. Thus ab remains inessential (essential).

For example, suppose ab is inessential and in particular γ_{ab} is inessential. Fix a and perturb b to b' along γ_{ba}. As $(\gamma_{ab}, \overline{ab})$ is inessential, there is a disk D bounded by $\overline{ab} \cup \gamma_{ab}$ such that K does not intersect the interior of D. Now to D, add the union of chords $\bigcup_p \overline{ap}$ where $p \in \gamma_{bb'}$. This creates a new disk D' bounded by $\overline{ab'} \cup \gamma_{ab'}$. Provided $\overline{ap} \cap K \neq \emptyset$

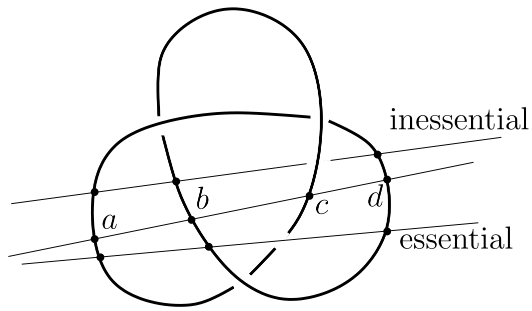

Figure 3.4: Secant bd changes from inessential to essential through trisecant bcd and so trisecant abd changes from inessential to essential through quadrisecant $abcd$.

for $p \in \gamma_{bb'}$, then K does not intersect the new disk D' and $\gamma_{ab'}$ and secant ab' remain inessential. A modification of this argument may be used for fixing b and perturbing a and for essential secants.

Thus all secants near secant ab will have the same topological type in the knotted Θ-graph *unless* K intersects $\overline{ab}$. In this case, $ab \in \pi_{13}(\mathcal{T})$. Thus secants change from inessential to essential *only* when there is a trisecant $t \in \pi_{13}(\mathcal{T})$ and each connected component of $S \setminus \pi_{13}(\mathcal{T})$ consists of either essential or inessential secants. (For example, Figure 3.4 shows a trisecant abd changing from inessential to essential through quadrisecant $abcd$. Here secant bd changes from inessential to essential via trisecant bcd.)

Moreover, ES is closed in S. Take a sequence of essential secants $\{s_i\}_{i=1}^{\infty}$, and suppose $s_i \to s$ where $s \in \pi_{13}(\mathcal{T})$. First assume $s_i \notin \pi_{13}(\mathcal{T})$. Then for any $\epsilon > 0$, far enough along the sequence each s_i is an ϵ-perturbation of s (in a C^1 sense). Thus by definition s is essential. Now assume the s_i are in $\pi_{13}(\mathcal{T})$. Then for each s_i there is a sequence $s_i^j \to s_i$, where s_i^j are essential secants and $s_i^j \notin \pi_{13}(\mathcal{T})$. For each i, there is a j_i such that the sequence $\{s_i^{j_i}\}_i$ of essential secants (not in $\pi_{13}(\mathcal{T})$) converges in C^1 to s. Again, for any $\epsilon > 0$, far enough along the sequence each $s_i^{j_i}$ is an ϵ-perturbation of s and thus by definition s is essential. □

Corollary 3.1.10. *In S, intervals of $\pi_{13}(\mathcal{T})$ between two connected components of essential secants are essential. Intervals of $\pi_{13}(\mathcal{T})$ between a component of essential and a component of inessential secants are essential.*

Proof. Apply Lemma 3.1.9. □

Note that Lemma 3.1.9 says nothing about intervals of $\pi_{13}(\mathcal{T})$ between two connected components of inessential secants. For non-generic knots it is possible to have such isolated essential secants, but this case does not matter for us.

We use this information to describe exactly how the sets of trisecants and essential trisecants are related. Let $\overline{\mathcal{ET}}$ denote the closure of $\mathcal{ET}$ in K^3 and let $\overline{ET} = \pi_{12}(\overline{\mathcal{ET}})$. In a similar way let $\overline{ET}^s = \pi_{12}(\overline{\mathcal{ET}}^s)$ and $\overline{ET}^d = \pi_{12}(\overline{\mathcal{ET}}^d)$.

Lemma 3.1.11. *Let K be a generic polygonal knot. Then $\mathcal{ET}$ is a closed subset of $\mathcal{T}$ and moreover, $\overline{\mathcal{ET}}$ is a finite union of closed circles and closed intervals in $\overline{\mathcal{T}}$.*

Proof. Trisecant abc is inessential when the second segment bc is essential. By definition, $\mathcal{ET} = \pi_{23}^{-1}(ES) \cap \mathcal{T}$. Now ES is closed in S and $\pi_{23}^{-1}(ES)$ is closed in K^3. Thus $\mathcal{ET}$ is a closed subset of $\mathcal{T}$. Lemma 3.1.9 showed connected components of $S \setminus \pi_{13}(\mathcal{T})$ are either inessential and essential secants. Let E_i denote the connected components of essential secants in $S \setminus \pi_{13}(\mathcal{T})$. By examining where $\pi_{23}(\mathcal{T})$ lies in relation to these components, we may determine which trisecants are essential or inessential.

Consider subsets of S consisting of all secants between one edge e_a of K and another edge e_b. Let this subset be denoted $e_a \times e_b$. As K is a generic polygonal knot, in $e_a \times e_b$ intervals of $\pi_{13}(\mathcal{T})$ and $\pi_{23}(\mathcal{T})$ either do not intersect or intersect in one or two (distinct) points. (Assume these arcs intersect in the point xy. Then the second and third point of a trisecant axy is the same as the first and third points of a trisecant xby. Thus the point of intersection represents some quadrisecant $axby$. By modifying the arguments found in Lemma 2.3.8 and Remark 2.3.9, it is easy to show that none, one or two quadrisecants exist for edges e_a and e_b.)

Thus we may view when $\pi_{23}(\mathcal{T})$ crosses into and out of the components of essential secants E_i. In each $e_a \times e_b \subset K^2$, $\pi_{23}(\mathcal{T})$ either remains inessential or essential, or changes from inessential to essential in one or two distinct places. As E_i are connected and as $\pi_{23}(\mathcal{T})$

changes from inessential to essential at most twice within each $e_a \times e_b$, $\overline{\mathcal{ET}}$ is a *finite* union of closed circles and closed intervals in $\overline{\mathcal{T}}$. □

Recall from Lemma 3.1.8 that essential alternating quadrisecants occur precisely when ET^s and ET^d have common points in S. To prove this happens, we need to examine further the structure of ET in S.

Lemma 3.1.12. *Let K be a generic polygonal knotted curve. Projection π_{12} is a piecewise smooth immersion of $\mathcal{ET}$ to S. In $\overline{T}$, $\overline{ET}$ is a finite union of closed intervals and closed circles and in S, ET intersects itself in double points.*

Proof. Lemma 3.1.11 showed $\overline{\mathcal{ET}}$ is a finite union of intervals and circles in $\overline{\mathcal{T}}$. Hence π_{12} is again a piecewise smooth immersion of $\mathcal{ET}$ to S and $\pi_{12}(\overline{\mathcal{ET}})$ inherits the properties of T found in Lemmas 2.3.7, 2.3.8 and 2.3.10. □

Lemma 3.1.13. *Let K be a generic polygonal knotted curve. In S, $d(ET^d, \tilde{\Delta}_-) \geq h$ and $d(ET^s, \tilde{\Delta}_+) \geq h$, where h is the minimum edge length of K.*

Proof. As ET is a subset of T, Lemma 2.3.15 gives the result. □

Thus ET^s and ET^d can only intersect in the interior of S. As ET is closed in T, the boundary of ET will contain points other than $\overline{ET} \setminus ET$ and can be found anywhere in $S \cup \Delta_+ \cup \Delta_-$. The content of the next lemma is that the points in $\overline{ET} \setminus ET$ must occur on Δ_+ or Δ_-.

Lemma 3.1.14. *Any point of $\overline{\mathcal{ET}} \cap \partial\mathcal{T}$ projects to a boundary point of S (that is on Δ_- or Δ_+).*

Proof. In S, we know points of $\partial T = \pi_{12}(\partial \mathcal{T})$ occur when an interval of adjacent trisecants end on a degenerate trisecant (Lemma 2.3.1). From the discussion in Section 2.3, ∂T occurs in the interior of S when the first point of the trisecant is fixed and the last two points converge. Let the points of the trisecant lie on edges e_1, e_2 and e_3 in that order and let e_2

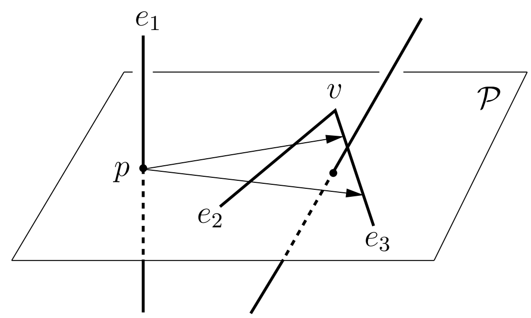

Figure 3.5: The interval of trisecants through pe_2e_3 must eventually be inessential as it ends at degenerate trisecant pvv. Close to vertex v the second segment through e_2e_3 is inessential. Trisecants through e_3e_2p may be essential until they end at degenerate trisecant vvp as the second segment e_2p may be essential.

and e_3 have common vertex v. Let e_1 intersect the plane $\mathcal{P}$ spanned by e_2 and e_3 at point p. See Figure 3.5. Eventually trisecants through pe_2e_3 must be inessential. This is because the second segment through e_2 and e_3 must be inessential. As the trisecant converges to pvv, there will be an embedded disk in $\mathcal{P}$ spanned by parts of e_2, e_3 and the part of the trisecant between e_2 and e_3.

On the other hand, the part of ∂T that occurs on Δ_+ or Δ_- might contain $\overline{ET}$. Use the same set up as in the previous paragraph and in Figure 3.5, but assume the points of the trisecant lie on $e_3e_2e_1$ in that order. The second segment e_2p of the trisecant may be essential. Thus there is nothing to prevent an interval of essential trisecants ending at degenerate trisecant vvp and $\pi_{12}(vvp) \in \Delta_+ \cup \Delta_-$. $\square$

Lemma 3.1.15. *Let K be a generic polygonal knotted curve. If $\overline{ET}^s \cap \overline{ET}^d \neq \emptyset$ in $\overline{S}$, then $ET^s \cap ET^d \neq \emptyset$ in S.*

Proof. If $\overline{ET}^s \cap \overline{ET}^d \neq \emptyset$, then either $ET^s \cap ET^d \neq \emptyset$ in S, or the intersection occurs in $\pi_{12}(\overline{\mathcal{ET}} \setminus \mathcal{ET})$. This is precisely the set $\pi_{12}(\overline{\mathcal{ET}} \cap \partial\mathcal{T})$. But from Lemma 3.1.14, points of $\pi_{12}(\overline{\mathcal{ET}} \cap \partial\mathcal{T})$ occur in either Δ_+ or Δ_- and we know from Lemma 3.1.13 that $\overline{ET}^s$ and $\overline{ET}^d$ can only intersect in the interior of S. Hence $ET^s \cap ET^d \neq \emptyset$ in S. $\square$

As $\overline{ET}^s$ and $\overline{ET}^d$ are closed compact subsets of a compact metric space $\overline{S}$, we may apply

Lemma 2.3.6 to them. Hence $d(ET^s, ET^d) = d(\overline{ET}^s, \overline{ET}^d) = 0$ if and only if $\overline{ET}^s \cap \overline{ET}^d \neq \emptyset$. Combining Lemma 3.1.8, Lemma 3.1.13 and Lemma 3.1.15, we see that in order to show that an essential alternating quadrisecant exists, it is sufficient to show $\overline{ET}^s$ and $\overline{ET}^d$ have common points in S. We will do this in the next section.

3.2 Essential alternating quadrisecants for generic polygonal knots

Lemma 3.1.15 showed that is is sufficient to show $\overline{ET}^s \cap \overline{ET}^d \neq \emptyset$ in S in order to prove that an essential alternating quadrisecant exists for generic polygonal knotted curves.

The underlying motivation for the proofs in this section is the idea that every curve going from left to right across a square is intersected by every curve going from the top to the bottom. In our case, the set of secants S is not a square but an annulus. This may be thought of as a square with the left and right edges identified. Thus the intuition is that every curve winding once around the annulus is intersected by every curve going across (from top to bottom).

In Lemma 3.2.2 we show that all curves winding once around S (defined below) must be intersected by ET. Thus intuitively ET "goes across" the annulus S. This, together with the fact that ET^s avoids the top of S and ET^d avoids the bottom of S, implies that $\overline{ET}^s \cap \overline{ET}^d \neq \emptyset$ in S.

Definition 3.2.1. A **curve winding once around** S is a simple closed curve $c : [0, 1] \to S$ which is homotopic to the generator 1 in $\pi_1(S) \equiv \mathbb{Z}$. See Figure 3.6.

Lemma 3.2.2 (Pannwitz). *Let K be a nontrivial generic polygonal knotted curve. Every closed curve winding once around S has nonempty intersection with the set of essential trisecants.*

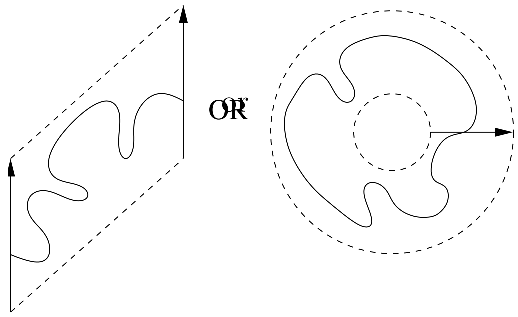

Figure 3.6: A curve winding once around S is a simple closed curve homotopic to the generator 1 in $\pi_1(S) \equiv \mathbb{Z}$.

Proof. Assume, by way of contradiction, that there is a curve c winding once around S with **empty** intersection with the set of essential trisecants. As c avoids the set of essential trisecants, each point of c is the first two points of a secant or an *inessential* trisecant. To simplify the argument we also assume that the curve c intersects $\pi_{12}(\mathcal{T})$ transversely away from self-intersections. As the set of essential secants is closed in $\overline{S}$, curves avoiding ET are in an open set. Thus any curve which is not transverse to $\pi_{12}(\mathcal{T})$ or which passes through a self-intersection of $\pi_{12}(\mathcal{T})$ has another curve arbitrarily close to it which is. Once we have proved the theorem for curves c which intersect $\pi_{12}(\mathcal{T})$ transversely away from self-intersections, we have proved it for all curves winding once around S.

Let us first assume each point of c is a secant. We will deal with the case where a point of c is the first two points of a trisecant later. (Such a point is in $\pi_{12}(\mathcal{T} \setminus \mathcal{ET})$.) Let $c = (x(s), y(s))$ where we have parameterized $[0, 1]$ with respect to arclength. Construct all (geodesic) rays $\overrightarrow{xy} \setminus \overline{xy}$. This is the collection of rays that start at $y(s)$ (the second point of the secant) and extend to infinity in the direction of the vector from $x(s)$ to $y(s)$. Such a ray is illustrated in Figure 3.7.

With the point at infinity, the union of the rays forms a spanning disk D with K as its boundary. This is because of the continuity of the curve c and the fact that c is a closed curve homotopic the generator 1 in $\pi_1(S)$. Thus $y(s)$ has degree 1 on the annulus S, hence $y(s)$ traverses the knot once. The assumption that each point on c is a secant means that the

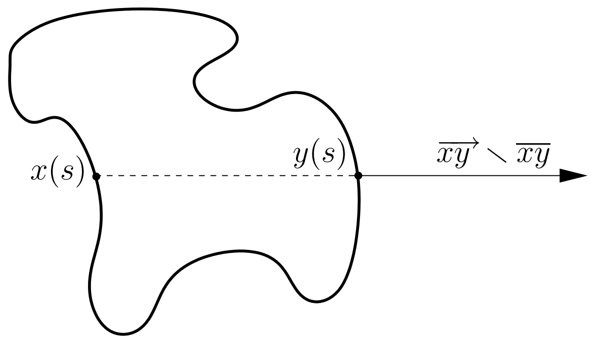

Figure 3.7: An example of a ray $\overrightarrow{xy} \setminus \overline{xy}$. The union of such rays and the point at infinity form a disk with the knot as its boundary.

rays do not intersect the knot again. However, the disk may have self-intersections. The rays may intersect each other away from the boundary. There may also be self-intersections on the boundary of the disk because $y(s)$ may not be strictly monotonic around K.

The aim is to apply Dehn's Lemma (Lemma 1.2.20). To do so, we must find a disk which is the image of a PL-map and whose self-intersections do not occur on the boundary. We do this by altering the disk D constructed above. Construct an ϵ-neighborhood N around the knot K. This is the set of all points within ϵ of the knot. We take ϵ small enough so that this a regular neighborhood (one such that K is a strong deformation retract of N). If necessary perturb N slightly so that D intersects the boundary of N, ∂N, transversely. The disk D intersects ∂N in several kinds of closed curves. Most are homologous to 0 on ∂N, but one will be homologous to a curve γ of homotopy class $(1, m)$ (as $y(s)$ has degree 1 on the knot). We wish to replace D with a disk that only intersects ∂N in γ.

The curves homologous to 0 on ∂N are disjoint, thus abstractly, these intersection curves are many families of nested circles. To remove the curves homologous to 0, take the outermost intersection circle of a family of nested circles which has some points in its interior lying inside N. The circle bounds a disk A on D and a disk B on ∂N. Replace A by B and push B slightly off ∂N to simplify the intersection. Repeating this process for all other families of nested circles eliminates all curves homologous to 0. Call the new disk so constructed D'. Then D' only intersects ∂N in the curve γ and no part of D' outside ∂N intersects D'

inside ∂N.

We wish to replace the part of D' outside ∂N with a PL disk. Approximate all of D' by a PL disk and use this to approximate γ very closely by a PL curve γ_0 outside ∂N. (Note γ_0 is of the same homology class as γ in the solid toroidal shell they both lie on.) Remove the part of the PL disk inbetween γ_0 and K (this is mostly inside N). Thus we have a PL-disk with γ_0 as its boundary. We would like to join K to γ_0 with an embedded PL collar, but can't as γ_0 might have self intersections. Recall γ_0 is homologous to a curve of homotopy class $(1, m)$. Put another regular ϵ-neighborhood N_1 inside N. On ∂N_1 put an embedded $(1, m)$ curve denoted γ_1. Thus γ_0 and γ_1 live on the boundary of a solid toroidal shell where they are homologous. We use a PL-homotopy in this shell to join γ_0 and γ_1 and then join γ_1 to K by an embedded PL-collar. This creates a new disk with K as its boundary and with no boundary intersections. By Dehn's Lemma, we may replace this new disk by an embedded one, giving a contradiction to knottedness.

Now on the original curve c there may be a secant which is the first two points of a trisecant. We assumed that $\pi_{12}(\mathcal{T})$ intersects c transversely away from self-intersections, thus such a point is isolated and is in $\pi_{12}(\mathcal{T} \setminus \mathcal{ET})$, hence is the first two points of an inessential trisecant xyz. We wish to perform surgery on the disk D so that the third point z of the trisecant no longer intersects the disk. Note that the second segment $\overline{yz}$ of the trisecant is inessential. This means that segment $\overline{yz}$ and one arc of the knot spans a disk $\mathcal{D}$ whose interior avoids the knot. (See Figure 3.8 (a).) The spanning disk $\mathcal{D}$ intersects the disk D in the segment $\overline{yz}$. Take a small embedded ϵ-neighborhood of the knot, this intersects both the original disk D and the spanning disk $\mathcal{D}$. Now make two copies of the spanning disk $\mathcal{D}$ and move them apart. Remove the parts of the copies of $\mathcal{D}$ inside the ϵ-neighborhood. Also remove the parts of D inside the two copies of $\mathcal{D}$ (this is a small strip about the inessential segment $\overline{yz}$). (See Figure 3.8 (b).) After smoothing things out, the original disk D has been altered so that it does not intersect the third point of the trisecant. However the new disk does intersects itself near the second point y of the trisecant. This is illustrated in

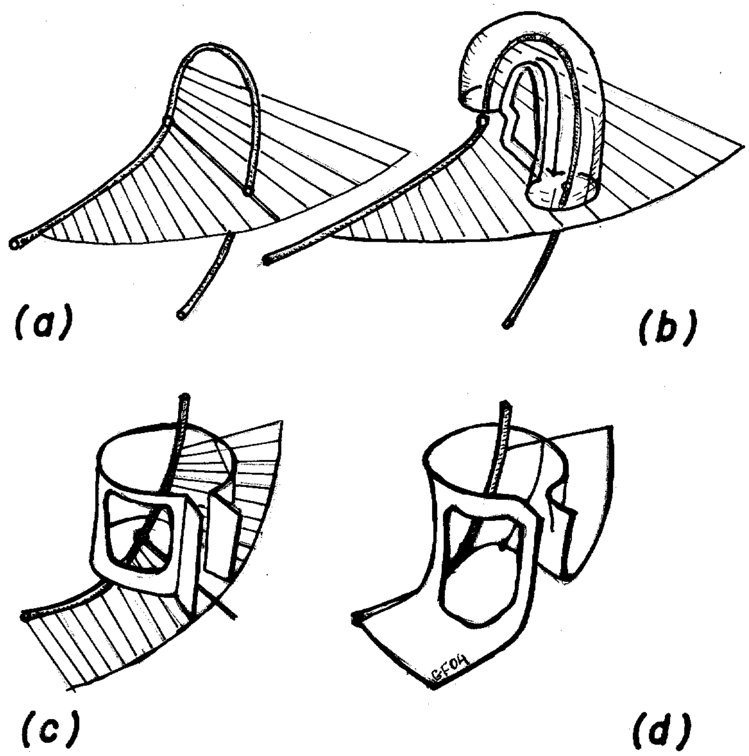

Figure 3.8: Figure illustrating the surgery described in Lemma 3.2.2.

Figure 3.8 (c) and (d). This surgery may be repeated for any other points of c which are the first two points of an inessential trisecant. Thus we obtain a disk D whose interior avoids K and we apply Dehn's Lemma as before to get a contradiction to knottedness. □

Proposition 3.2.3. *Let A and B be two closed subsets of the closed annulus $\overline{S}$, such that A lies outside some neighborhood of Δ_+ and B lies outside some neighborhood of Δ_-. If $A \cap B = \emptyset$, then there is a curve winding once around S avoiding $A \cup B$.*

Proof. Look at the Mayer-Vietoris sequence:

$$\cdots \to H_1(S \smallsetminus (A \cup B)) \to H_1(S \smallsetminus A) \oplus H_1(S \smallsetminus B) \to H_1(S) \to \dots$$

Using the assumption that A lies outside some neighborhood U of Δ_+, we construct a path α in $U \subset S \smallsetminus A$ which winds once around S. Similarly, as B lies outside some neighborhood V of Δ_-, we construct a path β in $V \subset S \setminus B$ which winds once around S with reverse orientation. These paths represent homology classes in $H_1(S \setminus A)$ and $H_1(S \setminus B)$ and the image of these classes in $H_1(S)$ is homologous to +1 and -1 respectively. The map $f : H_1(S \setminus A) \oplus H_1(S \setminus B) \to H_1(S)$ takes a pair (p, q) to the sum of the image of p and the image of q in $H_1(S)$. Thus the image of (α, β) under f is 0. Now the map $g : H_1(S \setminus (A \cup B)) \to H_1(S \setminus A) \oplus H_1(S \setminus B)$ takes a class γ to the pair $(p, -q)$ where p is the image of γ in $H_1(S \setminus A)$ and q is the image of γ in $H_1(S \setminus B)$. Therefore, by exactness of the sequence at $H_1(S \setminus A) \oplus H_1(S \setminus B)$, we may find a γ in $H_1(S \setminus (A \cup B))$ which maps to (α, β). This γ may be represented by a path in $S \setminus (A \cup B)$. As γ maps to α, γ winds once around S and avoids $A \cup B$. □

Theorem 3.2.4. *Every nontrivial generic polygonal knotted curve in $\mathbb{R}^3$ has an essential alternating quadrisecant.*

Proof. Lemma 3.1.13 shows that Proposition 3.2.3 may be applied to $\overline{ET}^s$ and $\overline{ET}^d$ in S (which is a topological annulus). If $\overline{ET}^s \cap \overline{ET}^d = \emptyset$ in $\overline{S}$, then there exists a path winding once around S avoiding $\overline{ET}^s \cup \overline{ET}^d$. This is a contradiction to Lemma 3.2.2 (Pannwitz). Hence $\overline{ET}^s \cap \overline{ET}^d \neq \emptyset$ in S, and Lemma 3.1.15 implies ET^s and ET^s have common points in S, which by Lemma 3.1.8 shows that there is at least one essential alternating quadrisecant for a generic polygonal knotted curve. □

Chapter 4

Main Theorem and Corollaries

4.1 Main result

We are now in a position to extend Theorem 3.2.4 and use a limit argument to show that alternating quadrisecants exist for *all* nontrivial tame knots K in $\mathbb{R}^3$. First, we define the kind of closed curves and the kind of convergence that we need for the limit to make sense. Recall that we assume that a knotted Θ-graph is formed from three disjoint simple curves (denoted α, β and γ) from p to q.

Definition 4.1.1. By an **FTC curve**, we mean a curve which has finite total curvature. An **FTC knot** is a simple closed curve with finite total curvature. An **FTC theta graph** is a theta graph such that each curve α, β and γ has finite total curvature.

Definition 4.1.2. Suppose $\{K_i\}_{i=1}^{\infty}$ is a sequence of FTC (closed) curves. Then we say K_i **converges in a C^1 sense** to a limit curve K if there is a set of parameterizations of K_i such that for all $\epsilon > 0$ there is an I such that once $i > I$ then for all t, $K_i(t)$ is within ϵ of $K(t)$. Also for all t left and right unit tangent vectors of $K_i(t)$ and $K(t)$ are within ϵ of each other. Suppose $\{\Theta_i\}_{i=1}^{\infty}$ is a sequence of embedded FTC theta graphs. Then Θ_i **converges in a C^1 sense** to a limit curve Θ if each curve α_i, β_i and γ_i converges in a C^1 sense to α, β and γ respectively.

Theorem 4.1.3 (Main Theorem). *Every knotted curve in* $\mathbb{R}^3$ *has an alternating quadrisecant.*

Proof. This follows [Kup] but has been altered and extended to suit our case. Given a knotted curve K, there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K)$ is smooth. Furthermore h may be chosen so that h is smooth on $\mathbb{R}^3 \setminus K$ (see [Kup]). Use Lemma 4.1.4 to construct a sequence of generic polygonal knots $\{K_i\}_{i=1}^{\infty}$ such that K_i converges to K and with the following properties:

(1) $h(K_i) \cap h(K) = \emptyset$

(2) $h(K_i)$ is a section of the normal bundle of $h(K)$

(3) $h(K_i)$ converges in a C^1 sense to $h(K)$, the zero section.

By construction each generic polygonal K_i has the same isotopy type as K, thus each K_i is nontrivial. By Theorem 3.2.4 we know that each K_i has an essential alternating quadrisecant $a_ib_ic_id_i$. By picking a subsequence if necessary we may assume that the original sequence converges to $abcd \in K^4$. We also assume the order of $a_ib_ic_id_i$ along the knot is $a_ic_ib_id_i$. From Lemma 3.1.1, it is sufficient to show that b and c do not lie on the same straight subarc of K, in order to show $abcd$ is a quadrisecant. Note each K_i does not intersect the interior of the essential segment $\overline{b_ic_i}$. (If it did, then $a_ib_ic_id_i$ would be included in a quintisecant, in contradiction to genericity condition (G2) on page 28.)

Let S_i be the essential segment $\overline{b_ic_i}$ and suppose, by way of contradiction, that the S_i's converge to a point p on K. Let B be an open ball centered at $h(p)$ lying inside the embedded normal tubular neighborhood about $h(K)$. Eventually there is an i, such that $h(S_i)$ and an arc $h(\gamma_{S_i})$ of $h(K_i)$ with the same endpoints as $h(S_i)$ are both contained in the ball. For each point $s \in h(S_i)$ take the unique point $t \in h(K_i)$ lying in the same fiber of the normal bundle over $h(K)$. Take the union of line segments from s to t, as illustrated in Figure 4.1. This forms a spanning disk D of $h(\gamma_{S_i}) \cup h(S_i)$ whose interior is *not* intersected by $h(K_i)$. If it did, then $h(\gamma_{S_i})$ is not a section. (In Figure 4.1 we see $h(S_i)$ intersects disk D, also D has boundary

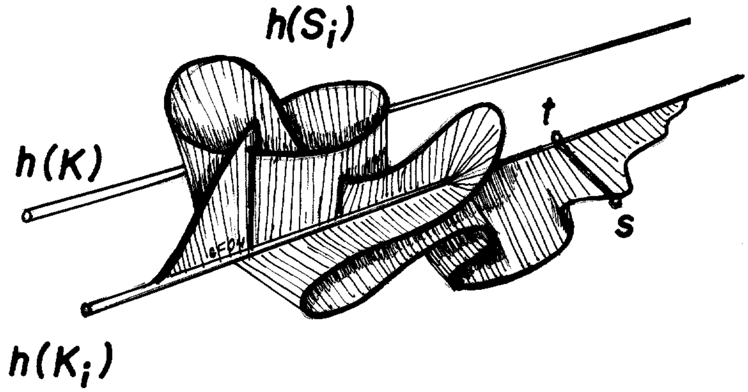

Figure 4.1: Disk constructed in the proof of Theorem 4.1.3. The boundary of D is $h(S_i) \cup h(\gamma_{S_i})$ and D is constructed as the union of line segments from $h(S_i)$ to $h(\gamma_{S_i})$.

intersections but all of these are allowed in the definition of inessential (see Definition 3.1.2).) Thus for the knotted theta $\Theta_i = h(K_i) \cup h(S_i)$, the pair $(h(\gamma_{S_i}), h(S_i))$ is inessential. Hence the pair (γ_{S_i}, S_i) is inessential, thus secant $b_i c_i$ is inessential, a contradiction.

We now show that b and c do not lie on the same straight subarc of K. The proof is similar to the previous argument. Suppose by way of contradiction that b and c lie on the same straight subarc of K. Again let $S = \overline{bc}$. Then $\gamma_{bc} = S$ thus is contained in K and $h(\gamma_{bc}) = h(S) \subset h(K)$. Take a small neighborhood of $h(\gamma_{bc})$ that lies inside the embedded normal tubular neighborhood about K. As $h(\gamma_{bc}) = h(S)$, far enough down the sequence $h(S_i)$ and $h(\gamma_{S_i})$ both lie inside this neighborhood. Proceed as above and construct a spanning disk of $h(S_i) \cup h(\gamma_{S_i})$ whose interior avoids $h(K_i)$. As before this implies that S_i is an inessential secant, a contradiction. □

All that remains is to construct the desired sequence of generic polygonal knots $\{K_i\}_{i=1}^{\infty}$.

Lemma 4.1.4. *Let K be a knotted curve and let h be the homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ described in [Kup] such that $h(K)$ is smooth and h is smooth on $\mathbb{R}^3 \setminus K$. Then there is a sequence of generic polygonal knots $\{K_i\}_{i=1}^{\infty}$ such that K_i converges to K with the following properties:*

(1) $h(K_i) \cap h(K) = \emptyset$

(2) $h(K_i)$ is a section of the normal bundle of $h(K)$

(3) $h(K_i)$ converges in a C^1 sense to $h(K)$, the zero section.

Proof. As $h(K)$ is smooth, there is a sequence of smooth knots $\{K_i^s\}_{i=1}^{\infty}$ such that $h(K_i^s) \cap h(K) = \emptyset$, $h(K_i^s)$ is a smooth section of the normal bundle of $h(K)$ and $h(K_i^s)$ converges smoothly to $h(K)$, which is the zero section. We wish to find a sequence of generic polygonal knots $\{K_i\}_{i=1}^{\infty}$ with similar behavior.

By Proposition 1.2.14, for each smooth knot K_i^s, there is a family of inscribed polygons which are ambient isotopic to K_i^s. We may choose the family so that they converge in a C^1 sense to K_i^s. Now perturb each of the inscribed polygonal knots so they are generic polygonal knots. This will give a sequence $\{K_i^j\}_{j=1}^{\infty}$ of generic polygonal knots which converge K_i^s in a C^1 sense.

As h is smooth on $\mathbb{R}^3 \setminus K$, the image of the generic polygonal knots $h(K_i^j)$ will not necessarily be polygonal. However, for each i we may find a j_i such that $h(K_i) := h(K_i^{j_i})$ is sufficiently close to $h(K_i^s)$ so that it has the desired relationship to $h(K)$. We do this in the following way. For each i, $h(K_i^s) \cap h(K) = \emptyset$. Hence there is some $\epsilon_i > 0$ such that $h(K_i^s)$ lies outside an open embedded normal tube of radius ϵ_i about $h(K)$. The map h is only smooth on the complement of K, so its derivatives may become unbounded at K (if K is not smooth for example). Thus, on the open set $\mathbb{R}^3 \setminus K$, the derivatives of h are unbounded. However, on any compact set (like the complement of an open normal embedded tube around K) a continuous map is bounded. Since a smooth map, like h, is one whose derivatives are all continuous, they are also all bounded by some constant C (on the compact set). As the derivatives of h are bounded outside the open normal embedded tube of radius ϵ_i, we choose a generic polygonal knot K_i so close to K_i^s that the difference in tangent vectors of $h(K_i)$ and $h(K_i^s)$ is very small (say ϵ_i/C). Then $\{K_i\}_{i=1}^{\infty}$ is the desired sequence of generic polygonal knots that converge to K such that for each i

(1) $h(K_i)$ is also a section of the normal bundle of $h(K)$

(2) $h(K_i) \cap h(K) = \emptyset$

(3) $h(K_i)$ converges in a C^1 sense to $h(K)$. □

In essence the Main Theorem showed that the limit of essential secants is still a secant. We now show (with some extra assumptions on the knotted curve K) that a limit of essential secants is still essential. (A version of this result may also be found in [DDS].)

We need to understand what happens to the notion of essential in such a limit. Recall that the property of a secant (or arc) of a knot being essential is a topological property, not of the knot, but of a knotted theta curve. Here we assume the knotted Θ-graph consists of a $C^{1,1}$ knot K and a secant segment $S = \overline{pq}$ (where $p, q \in K$). We are interested in a sequence of knotted thetas $\{\Theta_i\}_{i=1}^{\infty}$ consisting of generic polygonal knots K_i and secant segments $S_i = \overline{p_i q_i}$ (where $p_i, q_i \in K$). By assumption Θ_i converges to Θ and each $p_i q_i$ is essential. Thus if we can show that (far enough down the sequence) Θ_i is ambient isotopic to Θ, we will have shown that secant pq is also essential. As we are dealing with $C^{1,1}$ knots the notion of thickness (originally defined in [GM]) is useful.

Definition 4.1.5. Let K be a knotted curve. For any $p \in K$ the **thickness** $\tau(K)$ of K is defined in terms of the **local thickness** $\tau_p(K)$ at $p \in K$.

$$\tau(K) := \inf_{p \in K} \tau_p(K), \qquad \tau_p(K) := \inf_{\substack{q,r \in K \\ p \neq q \neq r \neq p}} r(p,q,r)\,.$$

Where, for any three distinct points p, q, r in $\mathbb{R}^3$, $r(p,q,r)$ is the radius of the (unique) circle through these three points. (Set $r = \infty$ if the points are collinear.)

Cantarella *et al* [CKS] also proved that for a C^1 knotted curve the thickness of K is the same as the **normal injectivity radius** of K. This is the largest r for which the union of open normal disks to K of radius r forms an embedded tube.

Proposition 4.1.6. *Let Θ be an embedded knotted theta graph consisting of a $C^{1,1}$ knot K and a secant-line $S = \overline{pq}$ (where $p, q \in K$). Let $\{\Theta_i\}_{i=1}^{\infty}$ be a sequence of embedded knotted* *theta graphs consisting of FTC knotted curves K_i and secant segments $S_i = \overline{p_i q_i}$ (where $p_i, q_i \in K_i$) such that Θ_i converges in a C^1 sense to Θ. Then for large enough i, Θ_i is ambient isotopic to Θ.*

Proof. The proof is inspired by both Milnor's [Mil] and Alexander and Bishop's [AB] proof of Proposition 1.2.18 that there are isotopic inscribed polygons for nontrivial knots of finite total curvature.

As both Θ_i and Θ are embedded theta graphs we know that K_i does not intersect the interior of S_i and similarly for K and S. By rescaling, we may assume our $C^{1,1}$ knot K has curvature bounded above by 1. By making small changes to Θ_i, using Euclidean similarities, we may assume that $p_i = p$ and $q_i = q$ (that is $S_i = S$). (Note that doing this will not change the C^1 convergence of the Θ_i. Also as the Euclidean similarities of Θ_i are ambient isotopic to the original Θ_i, so we don't change the conclusion of the proposition.)

About p and q, put a ball of radius δ, where δ is very small relative to the thickness τ of the knot K (say less than $\tau/100$). This choice of δ ensures just one arc of K is in $B_\delta(p)$ and that this arc of K won't turn very much at all before leaving the sphere. As S intersects K only at p and q, K leaves $B_\delta(p)$ and $B_\delta(q)$ at different points not on S. This is regardless of the tangent direction of K and S at p and q. Let $X = \gamma_{pq}$ and $Y = \gamma_{qp}$. Let p_S and q_S be the points where S leaves $B_\delta(p)$ and $B_\delta(q)$ respectively. Let p_X and q_X be the points where X leaves $B_\delta(p)$ and $B_\delta(q)$ respectively. Let p_Y and q_Y be the points where Y leaves $B_\delta(p)$ and $B_\delta(q)$ respectively. Figure 4.2 illustrates all of these points on $\partial B_\delta(p)$.

Outside the balls of radius δ about p and q, the knotted theta has been separated into three compact intervals ($\gamma_{p_X q_X}, \overline{p_S q_S}, \gamma_{q_Y p_Y}$). Let m be the minimum distance between each of these compact intervals. Let d be the minimum of m and the thickness τ of the knot K. Put a normal embedded tube of radius r less than $d/2$ about $\gamma_{p_X q_X}$ and $\gamma_{q_Y p_Y}$. By construction they do not intersect. As the K_i converge in a C^1 sense to K, the K_i eventually lie inside the normal tubes and behave as sections of the normal bundle, with K the zero section. (As the K_i behave as sections, they are ambient isotopic to K.) By assumption we know $S_i = S$.

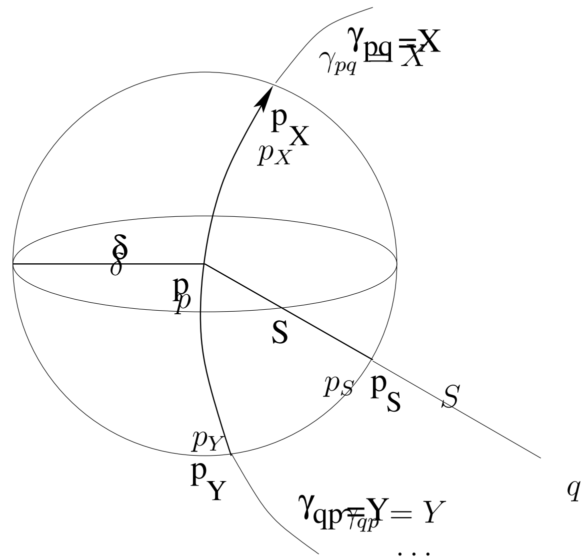

Figure 4.2: This figure shows the detail the points on $\partial B_\delta(p)$ described in Proposition 4.1.6.

Within $B_\delta(p)$ and $B_\delta(q)$ we find neighborhoods about K. These, as we shall see, are cones with vertex at p. The union of these cones and tubes defines a neighborhood about Θ_i and Θ. We define an isotopy on this neighborhood that moves Θ_i to Θ and vanishes on the boundary.

We construct the desired isotopy in three steps.

Step 1: Without loss of generality construct the isotopy on the tube about $\gamma_{p_X q_X}$. (We repeat the construction for the tube about $\gamma_{q_Y p_Y}$.) Extend the normal embedded tubes of radius r about $\gamma_{p_X q_X}$ so that it intersects $\partial B_\delta(p)$ and $\partial B_\delta(q)$ in topological disks. This is illustrated in Figure 4.3 (left) for the tube about $\gamma_{p_X q_X}$ intersecting $\partial B_\delta(p)$. Call this topological disk D. We foliate the tube about $\gamma_{p_X q_X}$ with topological disks transverse to K which after a short distance from D match the disks normal to K. Recall that we chose δ small enough that K does not turn very much at all within $B_\delta(p)$. Combine this with the fact that K is $C^{1,1}$ to see K is transverse to all the disks foliating the tube. As the K_i converge in a C^1 sense to K and behave as sections of the normal bundle, the K_i are also transverse to the disks. See Figure 4.3 (right). We construct an ambient isotopy on the tube as the union of isotopies on the disks. Each disk isotopy fixes the boundary, moves K_i to K

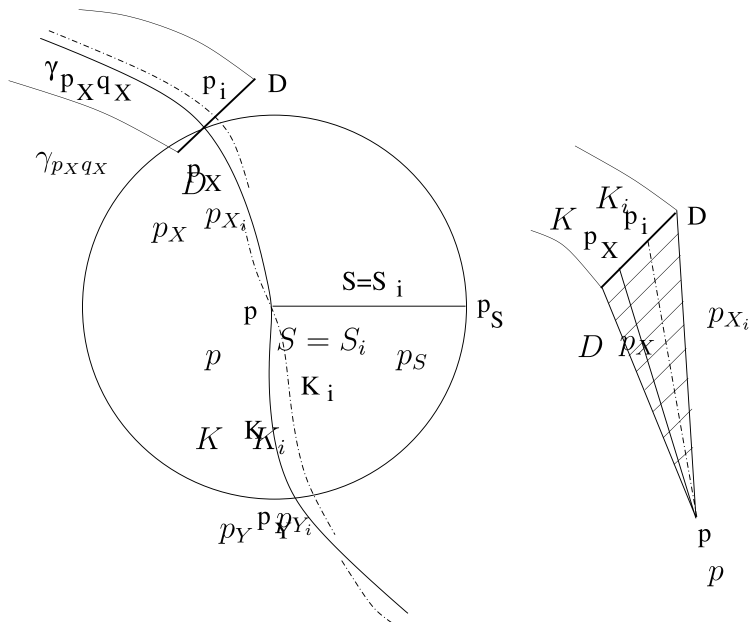

Figure 4.3: The left picture shows the tube about $\gamma_{p_Xq_X}$ intersecting $\partial B_\delta(p)$ in the disk D. Also that K_i intersects D in the point p_{X_i} (and similarly for p_{Y_i} near p_Y). The right picture shows the cone with vertex p and base D containing $\overline{pp_X}$ and $\overline{pp_{X_i}}$. The cone is foliated by the part of spheres centered at p inside the cone. The tube about $\gamma_{p_Xq_X}$ is foliated by topological disks which after a short distance match the normal disks to K.

and depends continuously on the points of K_i and K.

Step 2: Without loss of generality look at $B_\delta(p)$. (We repeat the construction for $B_\delta(q)$). Recall from Step 1 the topological disk D which is the intersection of the tube about $\gamma_{p_Xq_X}$ with $\partial B_\delta(p)$. Let K_i intersect D in the point p_{X_i} and let p_{Y_i} be the similar point near p_Y. These points are illustrated in Figure 4.3 (left). We aim to construct a cone with vertex p and base D which contains γ_{pp_X} and $\gamma_{pp_{X_i}}$. We won't be able to do this if K and/or K_i is tangent to $S = S_i$ at p. Thus first use Lemma 4.1.7 to construct an ambient isotopy supported in $B_\delta(p)$ which fixes $\partial B_\delta(p)$ and fixes p and straightens γ_{pp_X} to $\overline{pp_X}$ and $\gamma_{pp_{X_i}}$ to $\overline{pp_{X_i}}$, straightens γ_{pp_Y} to $\overline{pp_Y}$ and $\gamma_{pp_{Y_i}}$ to $\overline{pp_{Y_i}}$ and leaves $S = S_i$ unchanged.

We now construct a (topological) cone with vertex p, base D, axis $\overline{pp_X}$ and containing $\overline{pp_{X_i}}$. The radius r of D was chosen small relative to the distance between p_X and p_S. Thus far enough along the sequence the cone contains $\overline{pp_X}$ and $\overline{pp_{X_i}}$ but not $\overline{pp_S}$. We construct a similar cone about $\overline{pp_Y}$ and $\overline{pp_{Y_i}}$. Within each cone use an ambient isotopy to move

K_i to K and fix the boundary. Specifically, look at the cone about $\overline{pp_X}$. Foliate the cone by topological disks which are the intersection of the cone by spheres centered at p; these are illustrated in Figure 4.3 (right). The ambient isotopy is the union of disk isotopies on these disks. Each disk isotopy moves K_i to K, fixes the boundary and depends continuously on the points being moved.

Step 3: Far enough down the sequence the K_i lie in the union of cones and tubes about K. In Steps 1 and 2 we have defined cone and disk isotopies which moves K_i to K within this neighborhood, is the identity on the boundary and fixes $S = S_i$. (Note that the cone and disk isotopies were chosen to agree on D.) We may extend this to an ambient isotopy of all of $\mathbb{R}^3$. Thus we have shown that far enough down the sequence $\Theta_i = K_i \cup S_i$ is ambient isotopic to $\Theta = K \cup S$. □

Lemma 4.1.7. *Let B be a ball centered at p with three arcs α, β, γ starting at p and proceeding out to ∂B transversely to the nested spheres around p. Then there is an ambient isotopy on B which fixes p, fixes ∂B and straightens α, β and γ to radii. (In fact there is such a ball isotopy for n such arcs $(\alpha_1, \ldots, \alpha_n)$.)*

Proof. This proof used ideas from S.B. Alexander and R.L. Bishop's [AB] proof that there are isotopic inscribed polygons for nontrivial knots of finite total curvature. Curves of finite total curvature may have cusps where the curvature is π and there even may be cusps where one branch wraps infinitely often around the other. S.B. Alexander and R.L. Bishop used a cone isotopy to show that locally each branch of such a cusp may be straightened. We modify their argument to construct a ball isotopy where many such strands may be straightened.

To straighten our arcs, we comb the three strands inwards from ∂B, which we identify with the unit sphere. For $0 < r \leq 1$, let S_r denote the sphere of radius r centered at p and let α_r denote the point of intersection of the curve α with S_r. We define the ambient isotopy on B as the union of isotopies on S_r (and p is fixed). Thus as $t = 0$ we want the identity on all S_r and at $t = 1$ all curves have been straightened to radii. At times in between we use

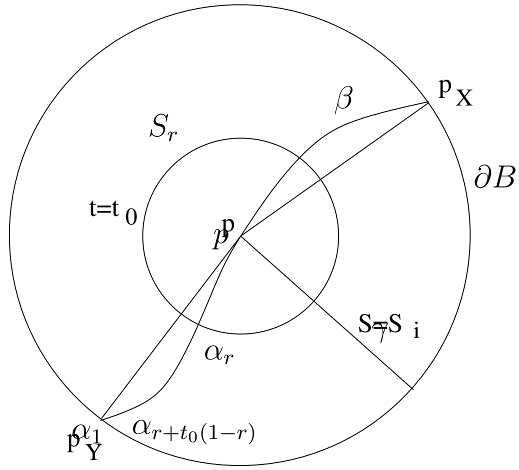

Figure 4.4: The isotopy on B described in Lemma 4.1.7 is the union of isotopies defined on spheres of radius r centered at p. On S_r at $t = t_0$ the points of α, β, γ are moved to the radial projection onto S_r of the points of α, β, γ on $S_{r+t_0(1-r)}$. (This is illustrated for α).

the position of the curves themselves to determine how they should move in the isotopy.

More precisely, at $t = t_0$, S_r "looks like" $S_{r+t_0(1-r)}$ (the sphere of radius $r + t_0(1-r)$). We describe what this means for the point α_r of α on S_r. At $t = t_0$ the isotopy moves the point α_r to the radial projection of the point of $\alpha_{r+t_0(1-r)}$ onto S_r. (See Figure 4.4.) Thus at $t = t_0$ on S_r, we see the same picture that was initially on $S_{r+t_0(1-r)}$; thus at time 1 the strands are all straight. Note that when $r = 1$ the isotopy does nothing, ∂B is fixed. □

Note that if two strands (say α and β) leave p with a common tangent line, then they might twist around each other infinitely often. But outside any S_r in B, there is only finitely much twisting, so the isotopy constructed in the lemma has no trouble combing this out.

Remark 4.1.8. Proposition 4.1.6 may be generalized. The result holds not just for knotted Θ-graphs consisting of a $C^{1,1}$ knot and a secant segment, but for any knotted graph of finite total curvature. In the proof of Proposition 4.1.6 we saw that thickness gives a good notion of how close curves must be in order to be ambient isotopic. However, in the more general version found below and in [DDS], no attempt has been made to find an analogous notion. Instead the proof shows that sufficiently close graphs are ambient isotopic.

Theorem 4.1.9. *Suppose Γ is a knotted graph of finite total curvature. Then any Γ' which is sufficiently close to Γ in a C^1 sense is ambient isotopic to Γ.*

Proof. The proof is found in [DDS]. However, the reader should note that it uses similar ideas to the proof of Proposition 4.1.6, so we give an outline of the argument here. Find points p_j on Γ such that these points divide Γ into arcs α_k each of curvature less than $\pi/17$. Let δ be the minimum distance between any two arcs α_k which are not incident. Consider disjoint balls B_j of radius $\delta/17$ centered at each p_j. Each arc α_k leaving p_j proceeds monotonically outwards to the boundary of B_j (since its curvature is too small to double back) and B_j contains no other arcs (since δ was chosen small enough).

Note that $\Gamma \smallsetminus \bigcup B_j$ is a compact union of disjoint arcs $\beta_k \subset \alpha_k$. Let $\epsilon \leq 2\delta/17$ be the minimum distance between any of these arcs β_k. Now consider a tube T_k about each β_k foliated by disks as in Step 1 of the proof of Proposition 4.1.6. Similar arguments to those in the proof of Proposition 4.1.6 are then used to construct the isotopy for any Γ' inside the neighborhood made of the union of all balls B_j and all tubes T_k. □

Note that Theorem 4.1.9 also shows that FTC knots which are close in a C^1 sense are ambient isotopic. Suppose we now define two (FTC) knots K_1 and K_2 to be equivalent if they are close in a C^1 sense. Then Theorem 4.1.9 shows this definition of knot equivalence is the same as all the other definitions of knot equivalence found in Proposition 1.2.11.

Theorem 4.1.9 allows us to prove the following result:

Theorem 4.1.10. *If K is a nontrivial knot of finite total curvature in $\mathbb{R}^3$, then K has an essential alternating quadrisecant.*

Proof. By Theorem 4.1.3 we know that every knotted curve (including FTC knots) has an alternating quadrisecant *abcd*. We must show that secant *bc* is essential. By Lemma 4.1.11 we find a sequence of generic polygonal knots $\{K_i\}_{i=1}^{\infty}$ converging in C^1 sense to K. By Theorem 3.2.4 each K_i has an essential alternating quadrisecant $a_ib_ic_id_i$ and by taking a subsequence if necessary, we may assume that $a_ib_ic_id_i$ converges to *abcd*. Then $S_i = \overline{b_ic_i}$

converges in a C^1 sense to $S = \overline{bc}$. Thus $\Theta_i = S_i \cup K_i$ converges in a C^1 sense to $\Theta = S \cup K$. As K_i is a generic polygonal knot it is certainly an FTC knotted curve. Thus by Theorem 4.1.9, for large enough i, Θ_i is ambient isotopic to Θ. As $b_i c_i$ is an essential secant, then so is secant bc and quadrisecant $abcd$ is an essential alternating quadrisecant. □

Lemma 4.1.11. *Let K be a nontrivial knot of finite total curvature, then there is a sequence $\{K_i\}_{i=1}^{\infty}$ of generic polygonal knots which converges in a C^1 sense to K.*

Proof. As K is $C^{1,1}$ it is also of finite total curvature. By Proposition 1.2.18 (see [AB] and [Mil]) we may find a sequence of inscribed polygonal knots $\{K_i^p\}_{i=1}^{\infty}$ which are isotopic to K. We may also choose this sequence to converge in a C^1 sense to K. Given any $\epsilon > 0$ far enough along the sequence we find a K_i^P which is within $\epsilon/2$ of K in a C^1 sense. As generic polygonal knots are an open dense set in the set of all polygonal knots, by perturbing K_i^p slightly we find a generic polygonal knot K_i within $\epsilon/2$ of K_i^p in a C^1 sense. Then K_i is within ϵ of K in a C^1 sense. Thus we have found a sequence $\{K_i\}_{i=1}^{\infty}$ of generic polygonal knot converging in a C^1 sense to K. □

The techniques and results of this section are now applied to G. Kuperberg's results about quadrisecants found in [Kup]. In this paper, G. Kuperberg defined an essential quadrisecant slightly differently to our Definition 3.1.6.

Definition 4.1.12 (Kuperberg). An n-secant $a_1 a_2 \ldots a_n \in K^n$ is **essential** if it is essential in **each** secant $a_i a_{i+1}$. Thus quadrisecant $abcd$ is essential if secants ab, bc and cd are essential.

Instead of working with generic polygonal knots, G. Kuperberg worked with knots that are parameterized by a generic polynomial (called generic polynomial knots). In [Kup] he proved the following theorems.

Theorem 4.1.13. *Every nontrivial generic polynomial knot in $\mathbb{R}^3$ has an essential quadrisecant.*

Theorem 4.1.14. *Every knotted curve in $\mathbb{R}^3$ has a quadrisecant.*

A small modification of the proof of Lemma 4.1.11 allows us to show that there is a sequence of generic polynomial knots $\{K_i\}_{i=1}^{\infty}$ which converge in a C^1 sense to a nontrivial C^1 knot K. This allows us to prove the following theorem which is also found in [DDS].

Theorem 4.1.15. *Every nontrivial C^1 knot K in $\mathbb{R}^3$ has an essential quadrisecant. (Here we mean essential in Kuperberg's sense.)*

Proof. By Theorem 4.1.14 a C^1 knot has a quadrisecant $abcd$. We must show secants ab, bc and cd are essential. As in the proof of Theorem 4.1.10 we find a sequence of generic polynomial knots K_i converging in C^1 to K. By Theorem 4.1.13 each K_i has an essential quadrisecant $a_ib_ic_id_i$ and as before we may assume $a_ib_ic_id_i$ converges to $abcd$. Now repeat the proof of Theorem 4.1.10 for each of the secants ab, bc and cd in turn to show that they are essential. □

4.2 Corollaries to the Main Theorem

The Main Theorem has two immediate applications. It is used to give alternate proofs to two previously known theorems about the geometry of knotted curves. The existence of alternating quadrisecants for knotted curves captures the intuition that a space curve must loop around at least twice to become knotted. This intuition is also reflected in results about total curvature and second hull of knotted curves.

Recall from Section 1.2 that for smooth curves, the total curvature of a closed curve is the total change in the angle of the unit tangent vector or equivalently the length of the tangent indicatrix. In 1929, M. Fenchel [Fen] proved that the total curvature of a closed curve in $\mathbb{R}^3$ is greater than or equal to 2π, equality holding only for plane convex curves. In 1947, K. Borsuk [Bor] extended this result to $\mathbb{R}^n$ and also conjectured the following theorem:

Theorem 4.2.1. *A knotted curve in $\mathbb{R}^3$ has total curvature greater than 4π.*

This result was first proved around 1949 by both J.W. Milnor [Mil] and I. Fáry [Far]. It has since become known as the Fáry-Milnor theorem. In his proof, J.W. Milnor used the idea of bridge number[1], making the observation that for a knotted curve, there are planes in every direction which cut the knot at least four times. More recently in [CKKS], the theorem was proved using the existence of second hull (defined later) for a knotted curve. Here we give a new proof using the existence of alternating quadrisecants.

Proof. By Theorem 4.1.3 we know that a knotted curve K has an alternating quadrisecant. An alternating quadrisecant is an inscribed polygon in K. By the definition of total curvature (see Definition 1.2.15), the total curvature of an inscribed curve is less than or equal to the total curvature of the curve it is inscribed in. The total curvature of an alternating quadrisecant is 4π. Therefore $\kappa(K) \geq 4\pi$. To get a strict inequality, observe that a knot is not coplanar. By repeatedly adding vertices to an alternating quadrisecant it eventually becomes an inscribed polygon with four vertices non-coplanar, giving it, and hence the knot, total curvature strictly greater than 4π. (See Lemma 4.2.2 from [Mil] below.) □

Lemma 4.2.2. *Adding a new vertex to a closed polygon cannot decrease its total curvature. The total curvature must increase if the new vertex a_j and three adjacent vertices (a_{j-1}, a_{j+1}, a_{j+2}) are not coplanar.*

The convex hull of a connected set K in $\mathbb{R}^3$ is characterized by the fact that every plane through a point in the convex hull must intersect K. If K is a closed curve, then a generic plane must intersect K an even number of times, so the convex hull is the set of points through which every plane cuts K twice.

This idea was generalized in [CKKS] as follows:

Definition 4.2.3. Let K be a closed curve in $\mathbb{R}^3$. Its n**th hull** $h_n(K)$ is the set of points $p \in \mathbb{R}^3$ such that K cuts every plane P through p at least $2n$-times. If the intersections

[1] The bridge number $b(K)$ of a knot K is the minimum number of bridges (or overpasses) occurring in a diagram of the knot, where the minimum is taken over all possible diagrams of K.

are transverse, then (thinking of P as horizontal, and orienting K) there are equal numbers of upward and downward intersections. To handle non-transverse intersections, we again orient K and adopt the following conventions. First, if $K \subset P$, we say K cuts P twice (once in either direction). If $K \cap P$ has infinitely many components, then we say K cuts P infinitely often. Otherwise, each connected component of the intersection is preceded and followed by open arcs in K, with each lying to one side of P. An **upward intersection** will mean a component of $K \cap P$ preceded by an arc below P *or* followed by an arc above P. (Similarly, a **downward intersection** will mean a component preceded by an arc above P or followed by an arc below P.) A glancing intersection, preceded and followed by arcs on the same side of P, thus counts twice, as both an upward and a downward intersection.

We have the geometric intuition that a knot must travel twice around some point in space. In his proof of the Fáry-Milnor theorem, Milnor observed that for a knotted curve, there are planes in every direction which cut the knot four times. In fact, there are points through which every plane cuts a knotted curve four times. But these points are precisely the second hull. In other words:

Theorem 4.2.4. *A knotted curve K in $\mathbb{R}^3$ has nonempty second hull.*

This result was originally proved in [CKKS]. This paper conjectured the existence of alternating quadrisecants as another way of proving that the second hull of knotted curves is nonempty. Here we give that alternate proof.

Proof. By Theorem 4.1.3 we know that a knotted curve K has an alternating quadrisecant $abcd$. Let t be a point on the midsegment $\overline{bc}$. We claim that t is in the second hull.

Project the knot radially to the unit sphere about t. Let the points a, b be at the north pole (N) and c, d be at the south pole (S) of the sphere. As $abcd$ is an alternating quadrisecant we see that the projected knot visits the poles in the order $NSNS$. To show that t is in the second hull, it suffices to show that the knot projection intersects any great circle at least four times. The same conventions given in Definition 4.2.3 apply to counting intersections of K with great circles.

There are two cases. Either the great circles are meridians (passing through the north and south pole) or they are not. Suppose the great circles are not meridians. Then as the projected curve visits the poles in order $NSNS$, it must cut such a great circle at least four times. Now suppose the great circle is a meridian. We divide the projected knot into four arcs, each arc is the part of K between a north and south pole. If all of these arcs lie on the meridian then this counts as an infinite number of intersections. If one of these arcs deviates from the meridian, this counts as two intersections. Thus if two or more arcs leave the meridian, then there are at least four intersections. Suppose that only one arc deviates from the meridian. Such a curve stays in the plane determined by the meridian most of the time and goes to one side once. This gives a knot of bridge number 1, which by [Mil] is the unknot. Thus for a knotted curve, the projected curve intersects all great circles at least two times and the second hull is non-empty. □

Essential alternating quadrisecants may be used to improve the known lower bounds of the ropelength of knotted curves from 24 to 31.32 (see also [DDS]). We devote the final chapter to this application.

Chapter 5

Ropelength of Thick Knots

5.1 The Ropelength Problem

The ropelength problem asks to minimize the length of a knotted curve subject to maintaining an embedded tube of fixed radius around the curve; this is a mathematical model of tying the knot tight in rope of fixed thickness.

Recall from Definition 4.1.5 that the thickness $\tau(K)$ of a space curve K is defined to be the infimal radius $r(x, y, z)$ of circles through any three distinct points of K. It is known [CKS] that $\tau(K) = 0$ unless K is $C^{1,1}$, meaning that its tangent direction is a Lipschitz function of arclength. When K is C^1, we can define normal tubes around K, and then indeed $\tau(K)$ is the supremal radius of such a tube that remains embedded. Note that in the existing literature, thickness is often defined to be the diameter rather than the radius of this thick tube.

Definition 5.1.1. The **ropelength** $R(K)$ of a knotted curve K is the (scale-invariant) quotient of the length of K over its thickness $\tau(K)$.

As ropelength is semi-continuous even in the C^0 topology on closed curves, it is not hard to show [CKS] that any (tame) knot or link type has a ropelength minimizer.

J. Cantarella, R.B. Kusner and J.M. Sullivan [CKS] gave certain lower bounds for the ropelength of links, which are sharp in certain simple cases where each component of the link is planar. However, these examples are still the only known ropelength minimizers. Recent work [CFKSW] describes a much more complicated tight (ropelength-critical) configuration B_0 of the Borromean rings; although the somewhat different Gehring thickness was used there, B_0 should still be tight (and presumably minimizing) for the ordinary ropelength we consider here. Each component of B_0 is still planar; it seems significantly more difficult to describe explicitly the shape of any tight (ropelength-critical) knot.

Numerical experiments [Pie, Sul] suggest that the minimum ropelength for a trefoil is slightly less than 32.7, and that there is another tight trefoil with different symmetry and ropelength 37.5. The best lower bound in [CKS] was 21.45; this was improved by Y. Diao [Dia], who showed that any knot has ropelength at least 24.

In this chapter we describe joint work with Y.Diao and J.M. Sullivan (found in [DDS]) where we use the idea of quadrisecants to get better lower bounds for ropelength. From Theorem 4.1.15 we know that any $C^{1,1}$ knot has an essential quadrisecant. It turns out that essential quadrisecants are exactly what we need for our improved ropelength bounds. (Note that a curve arbitrarily close to a round circle, with ropelength thus near 2π, can have an inessential quadrisecant.) By comparing the orderings of the four points along the knot and along the quadrisecant line we find three types of quadrisecants (simple, flipped and alternating). For each of these three types we use geometric arguments to obtain a lower bound for the ropelength of the knot having a quadrisecant of that type. The worst of these three bounds is 27.87, and this becomes the bound in Theorem 5.6.5. Theorem 4.1.10 implies that any nontrivial $C^{1,1}$ knot has an essential quadrisecant of alternating type. This together with Theorem 5.6.4, shows that any nontrivial knot has ropelength at least 31.32.

5.2 Knots with unit thickness

Recall that a **knot** is an oriented simple closed curve K in $\mathbb{R}^3$. Any two points p and q on a knot K divide it into two complementary subarcs γ_{pq} and γ_{qp}. Here γ_{pq} is the arc from p

to q following the given orientation on K. If $x \in \gamma_{pq}$, we will sometimes write $\gamma_{pxq} = \gamma_{pq}$ to emphasize the order of points along K. We will use ℓ_{pq} to denote the length of γ_{pq}; by comparison $|p-q|$ denotes the distance from p to q in space, the length of segment $\overline{pq}$.

Because the ropelength problem is scale invariant, we find it most convenient to rescale any knot K to have thickness (at least) 1. This implies that K is a $C^{1,1}$ curve with curvature bounded above by 1.

For any point $x \in \mathbb{R}^3$ and $r > 0$, let $B_2(x)$ denote the open ball of radius 2 centered at x. The following lemma, about the local structure of a thick knot, is now standard. (Compare [Dia, Lem. 4] and [CKS, Lem. 5].)

Lemma 5.2.1. *Let K be a knot of unit thickness. If $p \in K$, then $B_2(p)$ contains a single unknotted arc (of length at most 2π) of K which cuts the spheres around p transversely. If pq is a secant of K with $|p-q| < 2$, then the ball of diameter $\overline{pq}$ intersects K in single unknotted arc (either γ_{pq} or γ_{qp}) whose length is at most $2\arcsin(|p-q|/2)$.*

Proof. If there were an arc of K tangent at some point c to one of the spheres around p within $B_2(p)$, then triples near (p, c, c) would have radius less than 1. For the second statement, if K had a third intersection point c with the sphere of diameter $\overline{pq}$, then $r(p, q, c) < 1$. The length bound comes from Schur's lemma. □

An immediate corollary is:

Corollary 5.2.2. *If K has unit thickness, $p, q \in K$ and $x \in \gamma_{pq}$ with $p, q \notin B_2(x)$ then the complementary arc γ_{qp} lies outside $B_2(x)$.* □

The following lemma should be compared to [Dia, Lem. 5] and [CKS, Lem. 4], but here we strengthen it slightly and give a new proof.

Lemma 5.2.3. *Let K be a knot of unit thickness. If $p \in K$, then the radial projection of $K \smallsetminus \{p\}$ to the sphere $\partial B_2(p)$ is 1-Lipschitz, i.e., it does not increase length.*

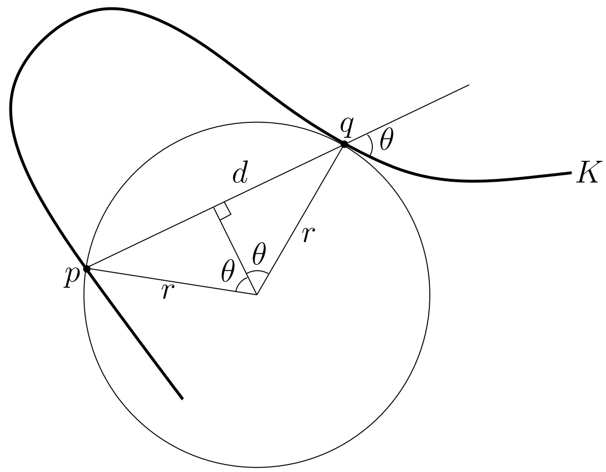

Figure 5.1: In the proof of Lemma 5.2.3, the circle tangent to K at q and passing through $p \in K$ has radius $r = d/(2\sin\theta)$ where $d = |p-q|$ and θ is the angle at q between K and $\overrightarrow{pq}$.

Proof. Consider what this projection does infinitesimally near a point $q \in K$. Let $d = |p-q|$ and let θ be the angle at q between K and the secant segment $\overline{pq}$. The projection stretches by a factor $2/d$ near q, but does not see the radial part of the tangent vector to K. Thus the local Lipschitz constant on K is $(2\sin\theta)/d$. Now consider the circle through p and tangent to K at q. Plane geometry (see Figure 5.1) shows that its radius is $r = d/(2\sin\theta)$. But it is a limit of circles through three points of K, so by the three-point characterization of thickness, r is at least 1; that is, the Lipschitz constant $1/r$ is at most 1. □

Corollary 5.2.4. *Suppose K has unit thickness, and $p, x, y \in K$ with $p \notin \gamma_{xy}$. Let $\angle xpy$ be the angle between the vectors $x-p$ and $y-p$. Then $\ell_{xy} \geq 2\angle xpy$. In particular, if xpy is a flipped trisecant in K, then $\ell_{xy} \geq 2\pi$.*

Proof. By the lemma, ℓ_{xy} is at least the length of the projected curve on $B_2(p)$, which in turn is at least the spherical distance $2\angle xpy$ between its endpoints. For a trisecant xpy we have $\angle xpy = \pi$, and $p \notin \gamma_{xy}$ exactly when the trisecant is flipped. □

Observe that a quadrisecant $abcd$ includes four trisecants: abc, abd, acd and bcd. Simple, flipped and alternating quadrisecants have different numbers of trisecants of different order.

We apply Corollary 5.2.4 to these trisecants to get lower bounds on ropelength for any curve with a quadrisecant.

Theorem 5.2.5. *The ropelength of a knot with a simple, flipped or alternating quadrisecant is at least* 2π*,* 4π *or* 6π*, respectively.*

Proof. Rescale K to have unit thickness, so its ropelength equals its length ℓ. Let $abcd$ be the quadrisecant, and orient K in the usual way, so that $b \in \gamma_{ad}$. In the case of a simple quadrisecant, the trisecant dba is of different order, so $\ell \geq \ell_{da} \geq 2\pi$, using Corollary 5.2.4. In the case of a flipped quadrisecant, the trisecants cba and bcd are of different order, so $\ell \geq \ell_{ca} + \ell_{bd} \geq 4\pi$. In the case of an alternating quadrisecant, the trisecants abc, bcd and dca are of different order, so $\ell \geq \ell_{ac} + \ell_{bd} + \ell_{da} \geq 6\pi$. □

Although G. Kuperberg [Kup] has shown that any nontrivial (tame) knot has a quadrisecant, and Theorem 4.1.3 shows that in fact is has an alternating quadrisecant, the bounds from Theorem 5.2.5 are not as good as the previously known bounds of [Dia] or even [CKS]. To improve our bounds, we will consider essential quadrisecants and the length of an essential arc of the knot in Section 5.5.

5.3 Length bounds in terms of segment lengths

Given a thick knot K with quadrisecant $abcd$, we can bound its length in terms of the distances along the quadrisecant line. Throughout this chapter, we will abbreviate these three distances as $r := |a-b|$, $s := |b-c|$ and $t := |c-d|$. We start with some lower bounds for r, s and t for quadrisecants of certain types.

Lemma 5.3.1. *If* $abcd$ *is a flipped quadrisecant of a knot with unit thickness, then the midsegment has length* $s \geq 2$*. Furthermore, if* $r \geq 2$ *then the whole arc* γ_{ca} *lies outside* $B_2(b)$*; similarly if* $t \geq 2$*, then* γ_{bd} *lies outside* $B_2(c)$*.*

Proof. Orient the knot in the usual way. If $s = |b - c| < 2$, then by Lemma 5.2.1 either $\ell_{cab} < \pi$ or $\ell_{bdc} < \pi$. But since cba and bcd are trisecants of different order, we have $\ell_{ca} \geq 2\pi$ and $\ell_{bd} \geq 2\pi$. This is a contradiction because $\ell_{cab} = \ell_{ca} + \ell_{ab}$ and $\ell_{bdc} = \ell_{bd} + \ell_{dc}$. The second statement follows from Corollary 5.2.2. □

Lemma 5.3.2. *If $abcd$ is an alternating quadrisecant of a knot of unit thickness, then $r \geq 2$ and $t \geq 2$. With the usual orientation, the entire arc γ_{da} lies outside of $B_2(b) \cup B_2(c)$. If $s \geq 2$ as well, then γ_{ac} lies outside of $B_2(b)$ and γ_{bd} lies outside of $B_2(c)$.*

Proof. If $r = |a - b| < 2$, then by Lemma 5.2.1 either $\ell_{acb} < \pi$ or $\ell_{bda} < \pi$. Similarly, if $t = |c - d| < 2$, then either $\ell_{cbd} < \pi$ or $\ell_{dac} < \pi$. But as in the proof of Theorem 5.2.5, we have $\ell_{ac} \geq 2\pi$ and $\ell_{bd} \geq 2\pi$, contradicting any of the inequalities above. Thus we have $r, t \geq 2$. Because a and d are outside $B_2(b)$ and $B_2(c)$, the remaining statements follows from Corollary 5.2.2. □

As suggested by the discussion above, we often find ourselves in the situation where we have an arc of a knot known to stay outside $B_2(x)$ — a ball of radius 2. We can compute exactly the minimum length of such an arc in terms of the following functions.

Definition 5.3.3. For $r \geq 2$, let $f(r) := \sqrt{r^2 - 4} + 2\arcsin(2/r)$. Given $r, s \geq 2$ and $\theta \in [0, \pi]$, the minimum length function m is defined by

$$m(r, s, \theta) := \begin{cases} \sqrt{r^2 + s^2 - 2rs\cos\theta} & \text{if } \theta \leq \arccos(2/r) + \arccos(2/s) \\ f(r) + f(s) + 2(\theta - \pi) & \text{if } \theta \geq \arccos(2/r) + \arccos(2/s) \end{cases}.$$

The function $f(r)$ will arise again in other situations. Here we note that it is a convex function of r, as we check by computing $f''(r) = 4r^{-2}(r^2 - 4)^{-1/2} > 0$.

The function m was defined exactly to make the following bound sharp:

Lemma 5.3.4. *Any arc γ from p to q staying outside $B_2(x)$ has length at least $m(|p-x|, |q-x|, \angle pxq)$.*

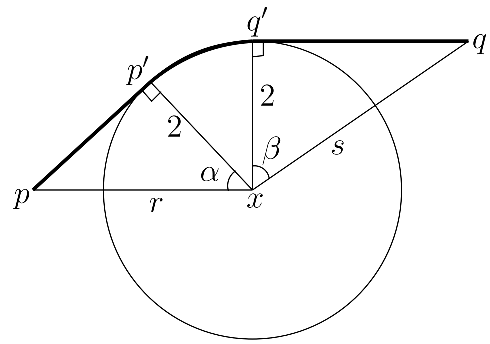

Figure 5.2: If points p and q are at distances $r, s \geq 2$ (respectively) from x, then the shortest curve from p to q avoiding $B_2(x)$ is planar. Either it is a straight segment or (in the case illustrated) it includes an arc of $\partial B_2(x)$. In either case, its length is at least $m(r, s, \angle pxq)$.

Proof. Let $r := |p-x|$, $s := |q-x|$ be the distances to x (with $r, s \geq 2$) and let $\theta := \angle pxq$ be the angle between $p-x$ and $q-x$. The shortest path from p to q staying outside $B := B_2(x)$ either is the straight segment or is the C^1 join of a straight segment from p to ∂B, a great-circle arc in ∂B, and a straight segment from ∂B to q. In either case, we see that the path lies in a plane through p, x, q. In this plane (shown in Figure 5.2) draw lines from p and q tangent to ∂B at p' and q', respectively. Let $\alpha := \angle pxp'$ and $\beta := \angle qxq'$, so that $\cos\alpha = 2/r$ and $\cos\beta = 2/s$. Clearly if $\alpha+\beta \geq \theta$ then the shortest path is the straight segment from p to q, with length $\sqrt{r^2+s^2-2rs\cos\theta}$. If $\alpha+\beta \leq \theta$ then the shortest path consists of the C^1 join described above, with length

$$\sqrt{r^2-4} + 2\big(\theta - (\alpha+\beta)\big) + \sqrt{s^2-4} = f(r) + f(s) + 2(\theta - \pi).$$

□

An important special case is when $\theta = \pi$. Here we are always in the case $\alpha + \beta \leq \theta$, so we get:

Corollary 5.3.5. *If p and q lie at distances r and s along opposite rays from x (so that $\angle pxq = \pi$), then the length of any arc from p to q avoiding $B_2(x)$ is at least*

$$f(r) + f(s) = \sqrt{r^2-4} + 2\arcsin(2/r) + \sqrt{s^2-4} + 2\arcsin(2/s)\,.$$

Lemma 5.3.6. *Let abcd be an alternating quadrisecant for a knot K of unit thickness (oriented in the usual way). Let $r := |a - b|$, $s := |b - c|$ and $t := |c - d|$ be the lengths of the segments along abcd. Then $\ell_{ad} \geq f(r) + s + f(t)$. The same holds if abcd is a simple quadrisecant as long as $r, t \geq 2$.*

Proof. In either case (and as we already noted in Lemma 5.3.2 for the alternating case), using Corollary 5.2.2 we find γ_{da} lies outside of $B_2(b) \cup B_2(c)$. As in the proof of Lemma 5.3.4, the shortest arc from d to a outside these balls will consist of the C^1 join of various pieces: these alternate between straight segments in space and great-circle arcs in the boundaries of the balls. Here, the straight segment in the middle always has length exactly $s := |b - c|$. As in Corollary 5.3.5, the overall length is then at least $f(r) + s + f(t)$ as desired. □

5.4 Arcs becoming essential

We have seen that the existence of a quadrisecant is not enough to get good lower bounds on ropelength. However, a certain class of *essential* quadrisecants give much better bounds. Our aim is to find the least length of an essential arc γ_{pq} for a knot K, and use this to get better lower bounds on the ropelength of knots. Recall that a knot together with a secant segment forms a knotted Θ-graph and that the notion of essential is a topological invariant of the (ambient isotopy) class of the knotted Θ. We repeat Definition 3.1.2 here as we will use it often in this section.

Definition 5.4.1. Suppose α, β and γ are three disjoint simple arcs from p to q, forming a knotted Θ-graph. Let $X := \mathbb{R}^3 \smallsetminus (\alpha \cup \gamma)$, and consider a parallel curve δ to $\alpha \cup \beta$ in X. Here by **parallel** we mean that $\alpha \cup \beta$ and δ cobound an annulus embedded in X as illustrated in Figure 5.3. We can choose the parallel so that δ is homologically trivial in X (that is, so that δ has linking number zero with $\alpha \cup \gamma$). Let $h(\alpha, \beta) \in \pi_1(X)$ denote the homotopy class of δ. Then (α, β) is **inessential** if $h(\alpha, \beta)$ is trivial. We say that (α, β) is **essential** if it is

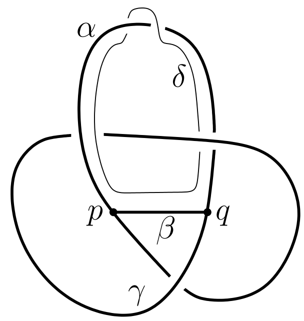

Figure 5.3: Essential arc of a knotted Θ-graph.

not inessential. Now let λ be a meridian loop (linking $\alpha \cup \gamma$) in the knot complement X. If the commutator $[\lambda, h(\alpha, \beta)]$ is nontrivial then we say (α, β) is **strongly essential**.

We now examine the properties of a strongly essential pair (α, β).

Lemma 5.4.2. *In a knotted* Θ*-graph* $\alpha \cup \beta \cup \gamma$*, the pair* (α, β) *is strongly essential if and only if* (γ, β) *is.*

Proof. Here we should be more careful about basepoints and orientations for the homotopy classes $h(\alpha, \beta)$ and $h(\gamma, \beta)$. So fix a basepoint near p and orient both these loops to follow β outwards and then return backwards along α or γ. Note that the product $h^{-1}(\alpha, \beta)h(\gamma, \beta)$ is a parallel to the knot $\alpha \cup \gamma$. Since a torus has abelian fundamental group, this parallel commutes with the meridian λ. It follows that $[\lambda, h(\alpha, \beta)] = [\lambda, h(\gamma, \beta)]$. □

Definition 5.4.3. The commutator $[\lambda, h(\alpha, \beta)]$ that comes up in the definition of strongly essential is also referred to as the **loop l_β around** β.

This is because $[\lambda, h(\alpha, \beta)]$ can be represented by a curve which follows a parallel β' of β, then loops around $\alpha \cup \gamma$ along a meridian near q, then follows β'^{-1}, then loops backwards along a meridian near p. (See Figure 5.4.)

In the proof of Lemma 5.4.2, we observed that $[\lambda, h(\alpha, \beta)]^{-1}[\lambda, h(\gamma, \beta)]$ is homotopic to a parallel of $\alpha \cup \gamma$. Let $X := \mathbb{R}^3 \smallsetminus (\alpha \cup \gamma)$ and suppose $\alpha \cup \gamma$ is knotted. Then curves parallel to $\alpha \cup \gamma$ are nontrivial in $\pi_1(X)$. Thus either $h(\alpha, \beta)$ and $h(\gamma, \beta)$ are both nontrivial

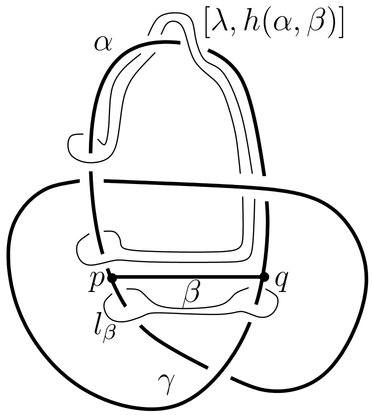

Figure 5.4: If λ is a meridian curve linking $\alpha \cup \gamma$, then the commutator $[\lambda, h(\alpha, \beta)]$ is homotopic to the loop l_β around β.

in $\pi_1(X)$, or one is trivial and the other nontrivial and homotopic to a parallel of $\alpha \cup \gamma$ in $\pi_1(X)$.

Lemma 5.4.4. *In a knotted* Θ*-graph* $\alpha \cup \beta \cup \gamma$*, if the pair* (α, β) *is strongly essential, then it and* (γ, β) *are essential.*

Proof. If (α, β) is strongly essential, then $[\lambda, h(\alpha, \beta)] = [\lambda, h(\gamma, \beta)]$ is a nontrivial element of $\pi_1(X)$, where $X := \mathbb{R}^3 \setminus (\alpha \cup \gamma)$. As λ is also nontrivial in $\pi_1(X)$, this implies $h(\alpha, \beta)$ and $h(\gamma, \beta)$ are nontrivial. □

We again apply the definition of (strongly) essential to arcs of a knot as follows.

Definition 5.4.5. If K is a knot and $p, q \in K$, let $S = \overline{pq}$. Assuming S has no interior intersections with K, we say that γ_{pq} is **(strongly) essential** for K if (γ_{pq}, S) is (strongly) essential in the knotted Θ-graph $K \cup S$. If S does intersect K, then we say γ_{pq} is **(strongly) essential** if for any $\epsilon > 0$ there is some ϵ-perturbation of S (in the C^1 sense, with endpoints fixed) to a curve S' such that $K \cup S'$ is an embedded Θ-graph in which (γ_{pq}, S') is (strongly) essential.

Note that Lemma 5.4.2 shows γ_{pq} is strongly essential if and only if γ_{qp} is also strongly essential. Recall the definition of an essential secant and G. Kuperberg's [Kup] definition of an essential n-secant.

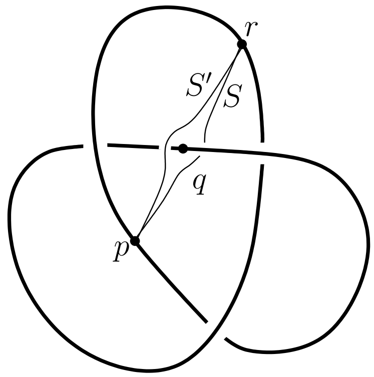

Figure 5.5: Secant pr is essential, but some nearby secants are not. By Theorem 5.4.7 there must be an essential trisecant pqr, because there are perturbations S and S' of $\overline{pr}$ which are essential and inessential, respectively.

Definition 5.4.6. A secant pq of K is **essential** if both subarcs γ_{pq} and γ_{qp} are essential. Otherwise it is **inessential**. A secant pq of K is **strongly essential** if γ_{pq} (equivalently γ_{qp}) is strongly essential. An n-secant $a_1a_2 \dots a_n$ is **essential** if each secant a_ia_{i+1} is essential. An n-secant is **inessential** if it is not essential.

Note that Lemma 5.4.4 shows that if secant pq of a knotted curve K is strongly essential then it is also essential.

We showed in Theorem 4.1.15 that every nontrivial $C^{1,1}$ knot has an essential quadrisecant. Our aim is to find the least length of an essential arc γ_{pq} for a knot K. We then use this to get better lower bounds on the ropelength of knots. This leads us to consider what happens when a lengthening arc becomes essential.

Theorem 5.4.7. *Let K be a knot with secant pr and suppose pr is in the boundary of the set of essential secants. (That is γ_{pr} is essential, but there are inessential arcs of K with endpoints arbitrarily close to p and r.) Then K must intersect the interior of segment $\overline{pr}$ and in fact there is some essential trisecant pqr.*

Proof. By Lemma 3.1.9, K intersects the interior of segment $\overline{pr}$ and so p and r are the first and last points of an n-secant $(n \geq 3)$ $pq_1 \dots q_{n-2}r$. We first assume that p and r are the

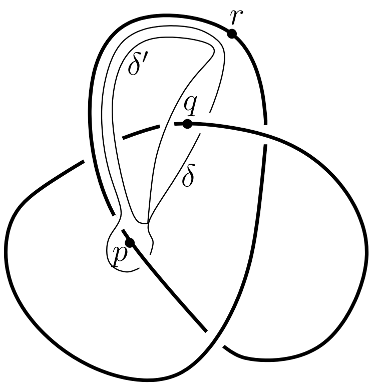

Figure 5.6: The left picture illustrates the loops $\delta = h(\gamma_{pr}, S)$ and $\delta' = h(\gamma_{pr}, S')$, where S and S' are the essential and inessential perturbations of $\overline{pr}$ as in Figure 5.5. The product $\delta^{-1}\delta'$ is homotopic to the loop $l_{\overline{pq}}$ around $\overline{pq}$. The middle picture shows an intermediate stage of the homotopy.

first and third points of a trisecant pqr and return to the other case at the end of the proof. We will show that pq and qr are strongly essential, hence pqr is an essential trisecant.

Let S and S' be two perturbations of $\overline{pr}$, with $K \cup S$ forming an essential Θ-graph and $K \cup S'$ an inessential one. (The first exists because γ_{pr} is essential, and the second because it is near inessential arcs.) These must differ merely by going to the two different sides of K near q. (See Figure 5.5.) Now, by the definition of essential, the homotopy classes $h(\gamma_{pr}, S)$ and $h(\gamma_{pr}, S')$ must differ: the first of these is nontrivial in $\pi_1(\mathbb{R}^3 \smallsetminus K)$ and the second is trivial.

The loops $h(\gamma_{pr}, S)$ and $h(\gamma_{pr}, S')$ each have linking number zero with K. Thus they differ not only by the meridian loop around K near q (changing S to S') but also by a meridian of K somewhere along the arc γ_{pr}, say near p. This means that the nontrivial homotopy class $h^{-1}(\gamma_{pr}, S)h(\gamma_{pr}, S')$ can in fact be represented by a loop $l_{\overline{pq}}$ around $\overline{pq}$. (See Figure 5.6.) Thus by definition, pq is strongly essential, as desired. By symmetry, the argument holds for qr as well.

We now repeat this calculation with basepoints involved. Let p', q', r' be points in $\mathbb{R}^3 \smallsetminus K$ near p, q, and r respectively. The points p, q, and r break the knot into three arcs γ_{pq}, γ_{qr} and γ_{rp}. Let $\alpha : p' \to q'$, $\beta : q' \to r'$ and $\gamma : r' \to p'$, be the respective three parallel curves to

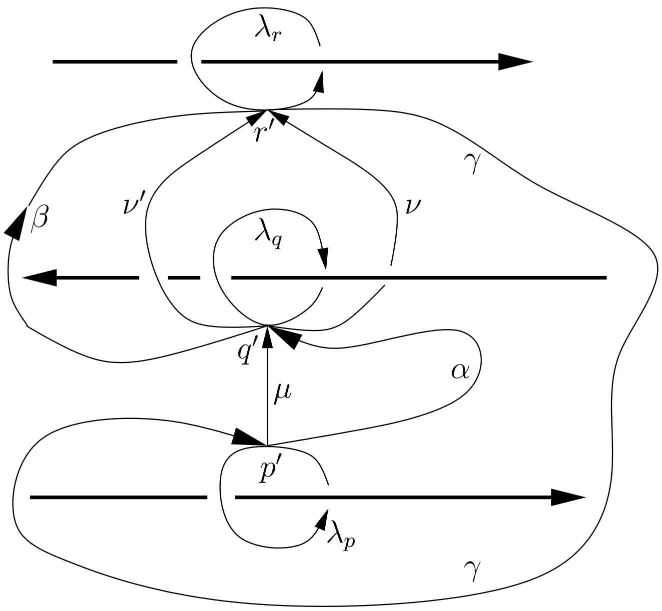

Figure 5.7: In the proof of Theorem 5.4.7, trisecant pqr has perturbed paths $S = \mu \cdot \nu$, $S' = \mu \cdot \nu'$, parallel paths α, β, γ, and meridian loops λ_p, λ_q, λ_r.

these arcs. Let $\mu : p' \to q'$ and $\nu : q' \to r'$ be curves parallel to trisecant line $\overline{pqr}$, chosen so that $\mu \cdot \nu = S$. Assume $\alpha \cdot \mu^{-1}$, $\beta \cdot \nu^{-1}$ and $\gamma \cdot \mu \cdot \nu$ have linking number 0 with K. Let λ_p, λ_q and λ_r be meridian loops around K at p', q' and r' respectively, each oriented to have linking number 1 with K. Figure 5.7 illustrates these curves and loops.

We chose these paths so that $h(\gamma_{pr}, S) = \alpha \cdot \beta \cdot \nu^{-1} \cdot \mu^{-1}$ and this and $\mu \cdot \nu \cdot \gamma$ are both nontrivial elements of $\pi_1(\mathbb{R}^3 \setminus K)$. Also recall that secant pq is essential if $\alpha \cdot \mu^{-1}$ and $\gamma^{-1} \cdot \beta^{-1} \cdot \mu^{-1}$ are both nontrivial in $\pi_1(\mathbb{R}^3 \setminus K)$ and is strongly essential if $[\mu \cdot \alpha^{-1}, \lambda_p]$ is nontrivial in $\pi_1(\mathbb{R}^3 \setminus K)$. Similarly secant qr is essential if $\beta \cdot \nu^{-1}$ and $\alpha^{-1} \cdot \gamma^{-1} \cdot \nu^{-1}$ are both nontrivial in $\pi_1(\mathbb{R}^3 \setminus K)$ and is strongly essential if $[\nu \cdot \beta^{-1}, \lambda_p]$ is nontrivial in $\pi_1(\mathbb{R}^3 \setminus K)$.

Again let S' be the perturbation of $\overline{pr}$ such that $h(\gamma_{pr}, S')$ is trivial in $\pi_1(\mathbb{R}^3 \setminus K)$. Now S' differs to S only by a loop about q. (See Figure 5.5.) Thus we may express the parallel curves to S' in terms of the others. Let ν', β' and γ' denote the parallel curves associated with the S'. (See Figure 5.7). The difference between the perturbed curve $S = \mu \cdot \nu$ and the perturbed curve $S' = \mu \cdot \nu'$ is $\nu' = \lambda_q \cdot \nu$, $\beta' = \lambda_q \cdot \beta$ and $\gamma' = \lambda_r^{-1} \cdot \gamma$. Note β' and γ' have been chosen so that $\beta' \cdot \nu'^{-1}$ and $\gamma' \cdot \mu \cdot \nu'^{-1}$ are have linking number 0 with K.

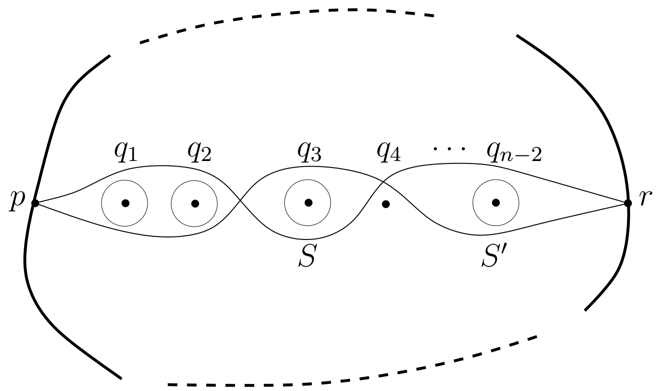

Figure 5.8: In the case where $\overline{pr}$ intersects K at many points q_i, it still has essential and inessential perturbations S and S' and these differ by loops about some subset of the q_i.

As before, the homotopy classes of $h(\gamma_{pr}, S)$ and $h(\gamma_{pr}, S')$ must differ: the first of these is nontrivial and the second is trivial in $\pi_1(\mathbb{R}^3 \setminus K)$. We use this information to show that secants pq and qr are strongly essential.

Compare loop $h(\gamma_{pr}, S')$ to $h(\gamma_{pr}, S)$. The loop $h(\gamma_{pr}, S')$ is given by $\alpha \cdot \beta' \cdot \nu'^{-1} \cdot \mu^{-1} = \alpha \cdot \lambda_q \cdot \beta \cdot \nu^{-1} \cdot \lambda_q^{-1} \cdot \mu^{-1} \sim \alpha \cdot \beta \cdot \nu^{-1} \cdot (\nu \cdot \beta^{-1} \cdot \lambda_q \cdot \beta \cdot \nu^{-1} \cdot \lambda_q^{-1}) \cdot \mu^{-1}$. But $\alpha \cdot \beta \cdot \nu^{-1} \cdot \mu^{-1}$ is the loop $h(\gamma_{pr}, S)$. These are not the same element in $\pi_1(\mathbb{R}^3 \setminus K)$. Thus $\nu \cdot \beta^{-1} \cdot \lambda_q \cdot \beta \cdot \nu^{-1} \cdot \lambda_q^{-1}$ is nontrivial. But $\nu \cdot \beta^{-1} \cdot \lambda_q \cdot \beta \cdot \nu^{-1} \cdot \lambda_q^{-1} = [\nu \cdot \beta^{-1}, \lambda_p]$, the relation which tells us whether secant qr is strongly essential or not. Thus qr is a strongly essential, hence essential secant.

Now repeat the above calculation, but recall $\lambda_q \sim \alpha^{-1} \cdot \lambda_p \cdot \alpha$. Then the loop $h(\gamma_{pr}, S')$ is given by $\alpha \cdot \beta' \cdot \nu'^{-1} \cdot \mu^{-1} = \alpha \cdot \lambda_q \cdot \beta \cdot \nu^{-1} \cdot \lambda_q^{-1} \cdot \mu^{-1} \sim \alpha \cdot \alpha^{-1} \cdot \lambda_p \cdot \alpha \cdot \beta \cdot \nu^{-1} \cdot \alpha^{-1} \cdot \lambda_p^{-1} \cdot \mu^{-1} \sim \lambda_p \cdot (\alpha \cdot \beta \cdot \nu^{-1} \cdot \mu^{-1}) \cdot (\mu \cdot \alpha^{-1} \lambda_p^{-1} \cdot \alpha \cdot \mu^{-1} \cdot \lambda_p) \cdot \lambda_p^{-1}$. But $\alpha \cdot \beta \cdot \nu^{-1} \cdot \mu^{-1}$ is the loop $h(\gamma_{pr}, S)$. Thus loops $h(\gamma_{pr}, S')$ and $h(\gamma_{pr}, S)$ are conjugate provided that $(\mu \cdot \alpha^{-1} \cdot \lambda_p^{-1} \cdot \alpha \cdot \mu^{-1} \cdot \lambda_p)$ is trivial in $\pi_1(\mathbb{R}^3 \setminus K)$. Note that $(\mu \cdot \alpha^{-1} \cdot \lambda_p^{-1} \cdot \alpha \cdot \mu^{-1} \cdot \lambda_p) = [\mu \cdot \alpha^{-1}, \lambda_p]$, the relation which tells us whether pq is strongly essential or not. Recall that if an element of a group is trivial, then a conjugate is also trivial. We have already noted the loops $h(\gamma_{pr}, S')$ and $h(\gamma_{pr}, S)$ are trivial and respectively nontrivial elements in $\pi_1(\mathbb{R}^3 \setminus K)$. This means $[\mu \cdot \alpha^{-1}, \lambda_p]$ is nontrivial and hence pq is a strongly essential, hence essential secant.

We first assumed that p and r are the first and third points of a trisecant pqr. We now assume that K intersects the interior of $\overline{pr}$ in more than one place. Write this n-secant as $pq_1q_2 \ldots q_{n-2}r$. We wish to show one of the trisecants pq_ir is essential.

From the definition of essential, we know there is a C^1 perturbation of $\overline{pr}$ to a path S such that (γ_{pr}, S) is essential. Again, since some nearby secants are inessential we can also find an inessential perturbation S'. Now S and S' differ by loops about some subset of the q_i. (See Figure 5.8.) We can start with S' and splice these loops in one-by-one, ending with S. At some stage (adding a loop around a particular q_i) the path first becomes essential. The argument given above then applies to show that pq_i and q_ir are strongly essential. (It is possible that some component of $K \cap \overline{pr}$ is an interval around q_i, but this does not affect our argument.) □

5.5 Minimum arclength for essential subarcs of a knot

We will improve the our previous ropelength bounds by first noting that a short arc must be inessential. A first bound is very easy:

Lemma 5.5.1. *If secant pq is essential in a knot of unit thickness then $|p - q| \geq 2$, and if arc γ_{pq} is essential then $\ell_{pq} \geq 2\pi$.*

Proof. If $|p - q| < 2$ then by Lemma 5.2.1 the ball B of diameter $\overline{pq}$ contains a single unknotted arc (say γ_{pq}) of K. Now for any perturbation S of $\overline{pq}$ which is disjoint from γ_{pq}, we can span $\gamma_{pq} \cup S$ by an embedded disk within B, whose interior is then disjoint from K. This means that γ_{pq} (and thus pq) is inessential.

Knowing that sufficiently short arcs starting at any given point p are inessential, consider now the shortest arc γ_{py} which is essential. From Theorem 5.4.7 there must be a trisecant pxy with both secants px and xy essential, implying by the first part that p and y are outside $B_2(x)$. Since px is essential, by the definition of y we have $x \notin \gamma_{ay}$, meaning that pxy is a trisecant of different order. From Corollary 5.2.4 we get $\ell_{pq} \geq \ell_{py} \geq 2\pi$. □

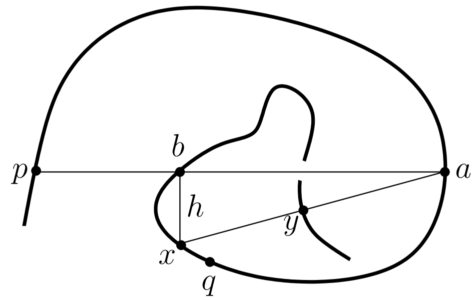

Figure 5.9: In the most intricate case in the proof of Lemma 5.5.2, we let γ_{pa} be the first essential arc from p, giving an essential trisecant pba. We then let γ_{ax} be the first essential arc from a, giving an essential trisecant ayx. Since $|y-x| \geq 2$ and $|a-y| \geq 2$, setting $h = |b-x|$ we have $|b-a| \geq 4-h$.

Lemma 5.5.2. *Let γ_{pq} be an essential arc of a unit thickness knot K, then either the length of γ_{pq} is at least $10\pi/3$ or there exists a trisecant pba of different order such that $a \in \gamma_{pq}$, $b \in \gamma_{qp}$ and $p, q \notin B_2(b)$.*

Proof. Assume that γ_{pq} is essential. Consider arcs starting from p and following the orientation of K. Let γ_{pa} be the *shortest* essential arc from p and observe that $q \notin \gamma_{pa}$. From Theorem 5.4.7 there must be a trisecant of different order pba with both secants pb and ba essential. Thus $|p-b| \geq 2$ and $|b-a| \geq 2$ and $\ell_{pq} \geq \ell_{pa} \geq 2\pi$. We conclude that $a, p \notin B_2(b)$ and also $\gamma_{pa} \notin B_2(b)$.

If $|q-b| \geq 2$, we are done. So let us assume that $|q-b| < 2$. In a similar manner, let γ_{ax} be the *shortest* essential arc from a, and ayx the trisecant of different order with both secants ay and yx essential. Note that $|a-x| \geq 4$. If $x \in \gamma_{aq}$, then we have $\ell_{pa} \geq 2\pi$ and $\ell_{aq} \geq \ell_{ax} \geq 2\pi$, so $\ell_{pq} \geq 4\pi > 10\pi/3$. Therefore we are also done in this case.

The only case that remains is if $x \in \gamma_{qb}$, as illustrated in Figure 5.9. Let $h := |b-x| \leq \ell_{xb}$ and note $h \in [0,2]$ since $q \in B_2(b)$. Since $|a-x| \geq 4$, we have $|b-a| \geq 4-h$, so $\ell_{pa} \geq \pi + f(4-h)$ by Corollary 5.3.5. On the other hand, since $|q-b| < 2$, the arclength $\ell_{qb} = \ell_{qx} + \ell_{xb} \leq \pi$ (by Lemma 5.2.1) and $\ell_{ax} = \ell_{aq} + \ell_{qx} \geq 2\pi$. Thus we have $\ell_{aq} \geq \pi + \ell_{xb} \geq \pi + h$. Hence $\ell_{pq} \geq 2\pi + f(4-h) + h$. It is not difficult to prove that the function on the right is an increasing function of $h \in [0,2]$. So its minimum is achieved

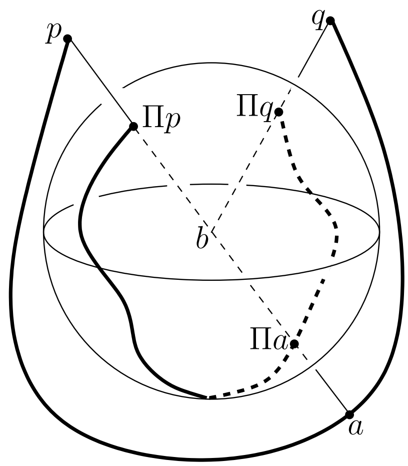

Figure 5.10: In the proof of Lemma 5.5.4, the projection of γ_{pq} to $\partial B_2(b)$ does not increase its length nor the distance between the endpoints. The projected curve includes antipodal points Πp and Πa, which bounds its length from below.

at $h = 0$, which yields a value $2\pi + f(4) = 7\pi/3 + 2\sqrt{3} > 10\pi/3$ as desired. $\square$

Definition 5.5.3. Define the function

$$g(r) := \begin{cases} 4\pi - 4\arcsin(r/4) & \text{if } 0 \le r \le 4, \\ 2\pi & \text{if } r \ge 4. \end{cases}$$

Note that $g(r)$ is continuous at $r = 4$. It is defined precisely so the following holds:

Lemma 5.5.4. *Let γ_{pq} be an essential arc in a knot K of unit thickness. Then $\ell_{pq} \ge g(|p-q|)$.*

Proof. Note that for $|p-q| \in [2,4]$ then $4\pi - 4\arcsin(|p-q|/4) \le 10\pi/3$. Since γ_{pq} is essential, from Lemma 5.5.2, either $\ell_{pq} \ge 10\pi/3 \ge 2\pi$ or there exists an essential trisecant pba of different order such that $a \in \gamma_{pq}$, $b \in \gamma_{qp}$ and $p, q \notin B_2(b)$. In the latter case, γ_{pq} stays outside $B_2(b)$. Let Π be the radial projection of $K \smallsetminus \{b\}$ to $\partial B_2(b)$. (See Figure 5.10.) From Lemma 5.2.3 this projection does not increase distance. Since $|\Pi p - \Pi q| \le |p-q|$ we have $4\pi - 4\arcsin(|\Pi p - \Pi q|/4) \ge 4\pi - 4\arcsin(|p-q|/4)$. Therefore it suffices to consider the case where $\gamma_{pq} \in \partial B_2(b)$. For two points $x, y \in \partial B_2(p)$, the shortest distance between x

and y is given by $d(x, y) = 4 \arcsin(|x - y|/4)$. Thus

$$\begin{aligned}\ell_{pq} &\geq \ell_{pa} + \ell_{aq} \geq 2\pi + d(a, q) \\ &= 2\pi + 4 \arcsin(|a - q|/4) = 2\pi + 4 \arccos(|p - q|/4) \\ &= 4\pi - 4 \arcsin(|p - q|/4).\end{aligned}$$

□

5.6 New lower bounds for ropelength

We now prove ropelength bounds for knots with different types of quadrisecants. The following lemma will be used repeatedly.

Lemma 5.6.1. *Recall that* $f(r) := \sqrt{r^2 - 4} + 2 \arcsin(2/r)$ *and that* $g(r) := 4\pi - 4 \arcsin(r/4)$ *for* $r \leq 4$ *while* $g(r) := 2\pi$ *for* $r \geq 4$. *The following functions of* $r \geq 2$ *have the minima indicated:*
The minimum of $f(r)$ *is* π *and occurs at* $r = 2$.
The minimum of $f(r) + g(r)$ *is* $7\pi/3 + 2\sqrt{3} > 10.79$ *and occurs at* $r = 4$.
The minimum of $g(r) + r$ *is* $2\pi + 4 > 10.28$ *and also occurs at* $r = 4$.
The minimum of $2f(r) + g(r) + r$ *is just over* 18.754 *and occurs for* $r \approx 2.006$.

Proof. Note that f is increasing, and g is constant for $r \geq 4$. Thus the minimum will occur in the range $2 \leq r \leq 4$, where $f' = \frac{1}{r}\sqrt{r^2 - 4}$ and $g' = -4/\sqrt{16 - r^2}$. Elementary calculations then give the results we want, where $r \approx 2.006$ is a polynomial root expressible in radicals. □

Theorem 5.6.2. *A knot with a essential simple quadrisecant has ropelength at least* $20\pi/3 + 4\sqrt{3} + 4 > 31.87$.

Proof. Rescale the knot K to have unit thickness, let $abcd$ be the quadrisecant and orient K in the usual way. Then the ropelength of K is $\ell_{ab} + \ell_{bc} + \ell_{cd} + \ell_{da}$. As before, let $r = |a - b|$, $s = |b - c|$ and $t = |c - d|$.

Lemma 5.5.4 bounds γ_{ab}, γ_{bc} and γ_{cd}. The quadrisecant is essential, so from Lemma 5.5.1 we have $r, s, t \geq 2$, and Lemma 5.3.6 may be applied to bound ℓ_{da}. Thus the ropelength of K is at least

$$\begin{aligned} & g(r) + g(s) + g(t) + \big(f(r) + s + f(t)\big) \\ = \quad & \big(g(r) + f(r)\big) + \big(g(s) + s\big) + \big(g(t) + f(t)\big). \end{aligned}$$

Since this is a sum of functions in the individual variables, we can minimize each term separately. These are the functions considered in Lemma 5.6.1, so the minima are achieved at $r = s = t = 4$. Adding the three values together, we find the ropelength of K is at least $20\pi/3 + 4\sqrt{3} + 4 > 31.87$. □

Theorem 5.6.3. *A knot with an essential flipped quadrisecant has ropelength at least* $20\pi/3 + 4\sqrt{3} > 27.87$.

Proof. Rescale K to have unit thickness, let $abcd$ be the quadrisecant, and orient K in the usual way. Then the ropelength of K is $\ell_{ab} + \ell_{bd} + \ell_{dc} + \ell_{ca}$. Again, let $r = |a - b|$, $s = |b - c|$ and $t = |c - d|$.

Since the quadrisecant is essential, from Lemma 5.5.1 and Lemma 5.3.1 we have $r, s, t \geq 2$. We apply Lemma 5.5.4 to γ_{ab} and γ_{dc} and Corollary 5.3.5 to γ_{bd} and γ_{ca}.

Thus the ropelength of K is at least

$$\begin{aligned} & g(r) + \big(f(r) + f(s)\big) + g(t) + \big(f(s) + f(t)\big) \\ = \quad & \big(g(r) + f(r)\big) + \big(2f(s)\big) + \big(g(t) + f(t)\big). \end{aligned}$$

Again, we can minimize in each variable separately using Lemma 5.6.1. We find the ropelength of K is at least $20\pi/3 + 4\sqrt{3} > 27.87$. ☐

Theorem 5.6.4. *A knot with an essential alternating quadrisecant has ropelength at least* 31.32.

Proof. Rescale K to have unit thickness, let *abcd* be the quadrisecant and orient K in the usual way. Then the ropelength of K is $\ell_{ac} + \ell_{cb} + \ell_{bd} + \ell_{da}$. Again, let $r = |a - b|$, $s = |b - c|$ and $t = |c - d|$.

The quadrisecant is essential, so from Lemma 5.5.1 and Lemma 5.3.2 we see $r, s, t \geq 2$. Thus Lemma 5.3.6 may be applied to γ_{da}. We apply Corollary 5.3.5 to γ_{ac} and γ_{bd}, and Lemma 5.5.4 to γ_{cb}.

We find that the ropelength of K is at least

$$\begin{aligned} & \big(f(r) + f(s)\big) + \big(f(s) + f(t)\big) + \big(g(s)\big) + \big(f(r) + s + f(t)\big) \\ = \quad & \big(2f(r)\big) + \big(2f(s) + g(s) + s\big) + \big(2f(t)\big). \end{aligned}$$

Again, we can minimize in each variable separately, using Lemma 5.6.1. Hence the ropelength of K is at least $4\pi + 18.754 > 31.32$. ☐

Theorem 5.6.5. *Any nontrivial knot has ropelength at least* 27.87.

Proof. Any knot of finite ropelength is $C^{1,1}$, so by Theorem 4.1.15 it has an essential quadrisecant. This must be either simple, alternating, or flipped, so one of the theorems above applies; we inherit the worst of the three bounds. ☐

Any knot of finite ropelength is $C^{1,1}$ and $C^{1,1}$ knots are certainly knots of finite total curvature. By Theorem 4.1.10, any nontrivial knot of finite total curvature has an essential *alternating* quadrisecant. Combining this with Theorem 5.6.4, we get an even better bound on the ropelength:

Theorem 5.6.6. *Any nontrivial knot has ropelength at least* 31.32.

We note that this bound is even better than the conjectured bound of 30.5 from [CKS, Conj. 26]. We also note that our bound cannot be sharp, for a curve which is C^1 at b cannot simultaneously achieve the bounds for ℓ_{cb} and ℓ_{bd} when $s \approx 2.006$. Probably a careful analysis based on the tangent directions at b and c could yield a slightly better bound. However, we note again that the numerical simulations have found trefoils with ropelength no more than 5% greater than our bound, so there is not much further room for improvement.

Appendix A

Computations

Assume we have a generic polygonal knotted curve K and that it is parameterized with respect to arclength. In Section 2.3 we examined the structure of the set of trisecants. Assume that the three points of trisecant $t \in \mathcal{T}$ lies in the interior of three edges e_1, e_2 and e_3 of K. Let E_1, E_2 and E_3 be the lines determined by e_1, e_2 and e_3. There are two cases considered in Lemma 2.3.1 and Lemma 2.3.2. Either the E_i are pairwise skew or two of the E_i are adjacent and the third is skew to both of these. In both cases we want to prove that along the interval of trisecants the points move smoothly along the corresponding edges.

We first assume that two of the E_i are adjacent and the third is skew to both of these as in Lemma 2.3.1. We give an explicit computation to show how the arclength along the adjacent edges are related. This in turn shows that along the interval of trisecants the points vary smoothly along the corresponding edges.

Assume that edges e_1 and e_2 are adjacent. Let $\mathcal{P}$ be the plane spanned by e_1 and e_2 and let p be the unique point of intersection of e_3 with $\mathcal{P}$. Without loss of generality, place x and y axes on $\mathcal{P}$ such that the point $p \in e_3$ is at the origin. Recall from Section 2.3 Figure 2.9 that $\mathcal{P}$ is separated into regions divided by lines E_1, E_2 and the line l between the non-intersecting vertices of e_1 and e_2. Trisecants occur when p lies in regions 1, 2 or 3 (and not the shaded regions). We first consider the case when p lies in regions 1 or 3 and trisecants are of the form pxy or xyp where $x \in e_1$ and $y \in e_2$.

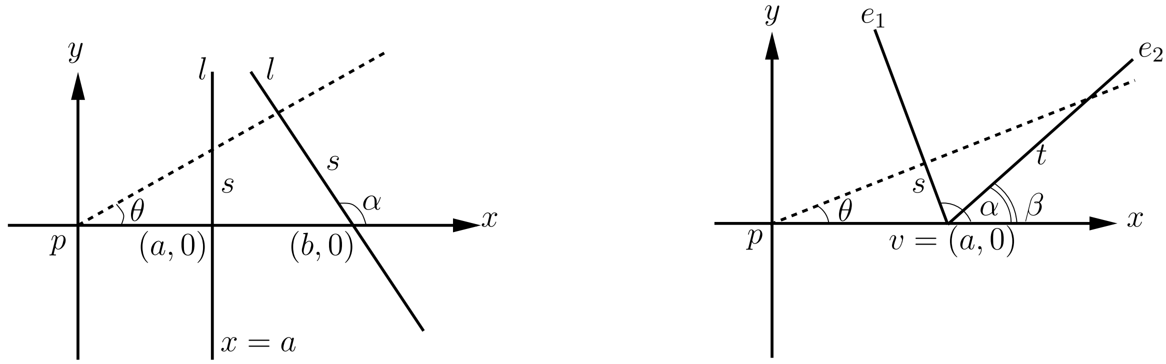

Figure A.1: Left picture shows how arclength s along a line l depends on the point of intersection with the x-axis, the angle α and the angle θ. The right picture shows how we may compare the arclength s and t along edges e_1 and e_2 respectively, as we move along the interval of trisecants.

Suppose edge e_1 determines the line $x = 1$. Let s be the arclength along e_1 (from the x-axis). Then $s = \tan\theta$, where θ is the angle from the x-axis to e_1. Now suppose e_1 determines the line $x = a$, then $s = a\tan\theta$. Now suppose e_1 determines the line which intersects the x-axis at b and which has angle α with the positive x-axis, as illustrated in Figure A.1 (left). Let s again be the arclength (from the x-axis) along the line. Then a short calculation reveals

$$s = \frac{b\tan\theta}{\sin\alpha - \cos\alpha\tan\theta}\,. \tag{A.1}$$

We are interested in how arclength changes along edges e_1 and e_2 as we move along the interval of trisecants. We have already assumed that the third point of the trisecant p is at the origin. Now place the x-axis to pass through the point p and the common vertex v of e_1 and e_2. Suppose v is at $(a, 0)$ and suppose e_1 and e_2 have angles α and β with the positive x-axis respectively. This is illustrated in Figure A.1 (right). (Note that angles α and β are never 0 or π — if they were it would contradict non-degeneracy.) Let s be the arclength (from the x-axis) along e_1 and t be the arclength (from the x-axis) along e_2. Equation A.1 shows that

$$s = \frac{a\tan\theta}{\sin\alpha - \cos\alpha\tan\theta}, \qquad t = \frac{a\tan\theta}{\sin\beta - \cos\beta\tan\theta}\,.$$

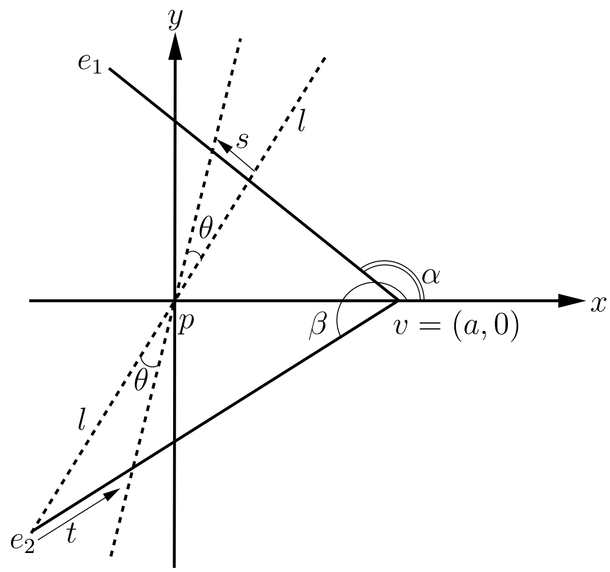

Figure A.2: Picture shows how we may compare the arclength s and t along edges e_1 and e_2 respectively, as we move along the interval of trisecants.

Now write $\tan\theta$ in terms of s and then substitute this into the equation above involving t. A short calculation shows

$$t = \frac{sa\sin\alpha}{a\sin\beta + s\sin(\beta - \alpha)}\,. \tag{A.2}$$

Suppose our interval of trisecants is of the form pxy (where $x \in e_1$ and $y \in e_2$). Suppose x moves along e_1 with unit speed, then from Equation A.2 we know how the corresponding point y of the trisecant moves along e_2. (The same argument hold for trisecants of the form xyp.) Thus along the interval of trisecants the corresponding points on e_1 and e_2 move smoothly and monotonically.

We may repeat the calculation for the situation where p lies in region 2 in Figure 2.9 and trisecants are of the form xpy where $x \in e_1$ and $y \in e_2$. Again let the x-axis pass through p and the common vertex v of edges e_1 and e_2. Let l be the line through p and the vertex of e_2 which is not v. The arclength s along e_1 is measured, not from the x-axis, but from the line l. The arclength t along e_2 is also measured from l. This is illustrated in Figure A.2. A short computation reveals that s and t are also related by Equation A.2. Thus along the interval

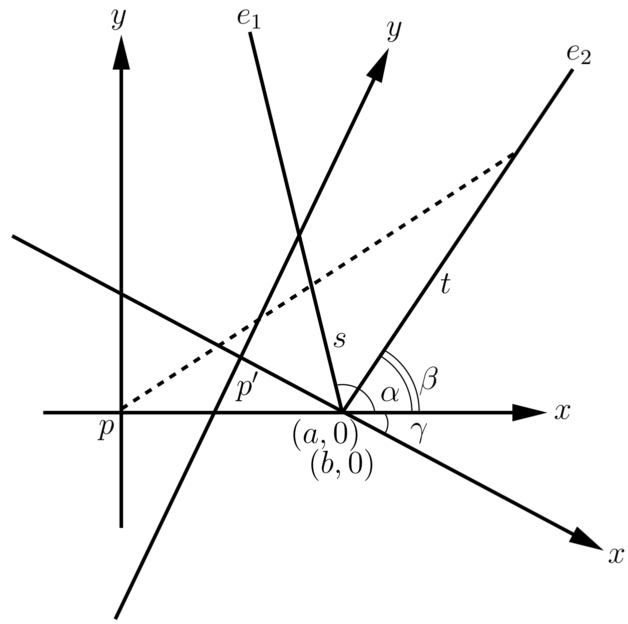

Figure A.3: We may compare the arclength s and t along edges e_1 and e_2 respectively, as we move along two different intervals of trisecants determined by p and p'.

of trisecants the corresponding points on edges e_1 and e_2 move smoothly and monotonically.

In S, the intervals of trisecants are not only smooth and monotonic, but they are conjectured to be transverse to each other at a point of intersection. We now give the computations which show this for Case 1 in the proof of Conjecture 2.3.13 (found on page 48). Here $\pi_{12}(t) = \pi_{12}(t')$ for $t = p_1p_2p$ and $t' = p_1p_2p'$, where p_1 and p_2 lie on adjacent edges e_1 and e_2 respectively, and p and p' lie on edges e_3 and e_3'. Let v denote the common vertex of e_1 and e_2 and let $\mathcal{P}$ denote the plane spanned by e_1 and e_2. Thus p and p' are the points where edges e_3 and e_3' intersect $\mathcal{P}$.

Let a be the distance from p to v and b be the distance from p' to v. Without loss of generality, assume $a > b$. In order for $\pi_{12}(t) = \pi_{12}(t')$, the point p' must lie in the interior of the triangle bounded by edge e_1, the line from p to v and the line from p to the other vertex of e_2 (i.e. not vertex v).

Construct xy-axes with p at the origin and so the x-axis passes through v as in Figure A.1 (right). Again, let α and β be the angles that e_1 and e_2 make with the positive x-axis respectively. On the same picture, construct a similar set of axes but with p' at the origin. This is illustrated in Figure A.3. We define angles $\hat{\alpha}$, $\hat{\beta}$ to be the angles that e_1 and e_2 make with the new x-axis, respectively. Let γ be the angle between the two x-axes. Note that $0 \leq \gamma \leq 180 - \alpha$. Then $\hat{\alpha} = \alpha + \gamma$ and $\hat{\beta} = \beta + \gamma$. As before, s and t denote the arclength along e_1 and e_2 respectively. Using equation A.2 we may write t either in terms of a, α, β or in terms of b, $\hat{\alpha} = \alpha + \gamma$, $\hat{\beta} = \beta + \gamma$.

$$t = \frac{sa\sin\alpha}{a\sin\beta + s\sin(\beta - \alpha)} = \frac{sb\sin(\alpha+\gamma)}{\sin(\beta+\gamma) + s\sin(\beta - \alpha)}.$$

Assume s changes with unit speed. In order to examine the tangent vectors of the two intervals of trisecants at $\pi_{12}(t) = \pi_{12}(t')$, we must understand dt/ds.

$$dt/ds = \frac{-a^2\sin\alpha\sin\beta}{(a\sin\beta + s\sin(\beta-\alpha))^2} = \frac{-b^2\sin(\alpha+\gamma)\sin(\beta+\gamma)}{(b\sin(\beta+\gamma) + s\sin(\beta-\alpha))^2}.$$

This is only equal for both intervals of trisecants if *both* $a\sin\alpha = b\sin(\alpha+\gamma)$ and $a\sin\beta = b\sin(\beta+\gamma)$. This can't happen as $a\sin x = b\sin(x+\gamma)$ only has one solution for $0 \leq x \leq \pi$. Thus dt/ds is different for both intervals of trisecants. Hence in S, the two intervals of trisecants have different tangent vectors and are transverse to each other at a point of intersection.

Finally, we assume that E_i are pairwise skew as in Lemma 2.3.2. These lines generate a doubly-ruled surface. We wish to show that along the interval of trisecants the points along the corresponding edges move smoothly and monotonically. We give an explicit computation for a particular doubly-ruled surface to show how the arclength along the edges are related. Take the doubly-ruled surface which is the hyperbolic paraboloid $z = y^2 - x^2$. Assume the point (a, b, c) lies on this surface. Then a short calculation shows that the vector equations of the two lines (one in each ruling) through (a, b, c) are

$$T_1(s) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ -2(b+a) \end{bmatrix}, \qquad E_1(t) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 2(b-a) \end{bmatrix}.$$

Now assume that $E_1(t)$ is the line determined by edge e_1 and $T_1(s)$ is a line which intersects E_1, E_2 and E_3. Suppose T_1 intersects E_2 when $s = s_0$, then the intersection of T_1 and E_2 is the point

$$T_1(s_0) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + s_0 \begin{bmatrix} 1 \\ -1 \\ -2(b+a) \end{bmatrix}.$$

The equation of line E_2 is thus

$$E_2(t) = T_1(s_0) + t \begin{bmatrix} 1 \\ 1 \\ 2(b-a-2s_0) \end{bmatrix}.$$

Now we move along the interval of trisecants through E_1, E_2, E_3. Say we start at trisecant line T_1 and end at trisecant line T_2. Trisecant line T_1 intersects line E_1 at $t = 0$ and trisecant line T_2 intersects line E_1 at $t = t_0$ say. (These lines are illustrated in Figure A.4.) The equation of T_2 is thus

$$T_2(s) = E_1(t_0) + s \begin{bmatrix} 1 \\ -1 \\ -2(b+a+2t_0) \end{bmatrix}.$$

We wish to compare the arclength along E_1 and E_2 inbetween T_1 and T_2. This is the same as measuring the length of vectors $\vec{v}_1$ and $\vec{v}_2$ (see Figure A.4).

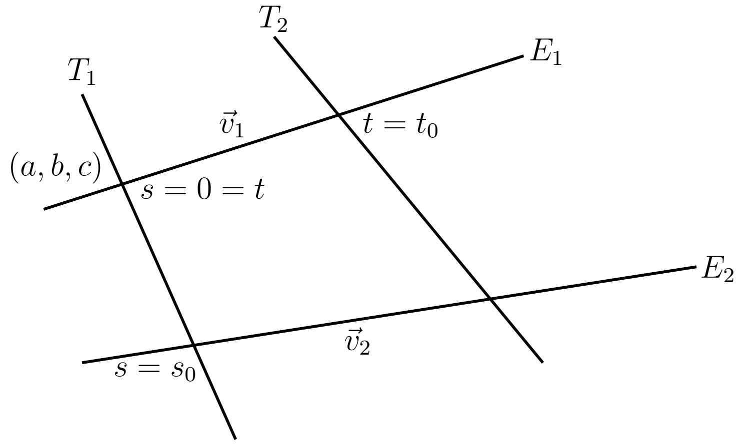

Figure A.4: Picture illustrates the relationship between the arclength along lines E_1 and E_2 as the trisecant lines move from T_1 to T_2.

First we find the point of intersection of $E_2(t)$ and $T_2(s)$. A short calculation shows this occurs when $s = s_0$ and $t = t_0$. This point may be represented in two ways:

$$T_1(s_0) + t_0 \begin{bmatrix} 1 \\ 1 \\ 2(b - a - 2s_0) \end{bmatrix} = E_1(t_0) + s_0 \begin{bmatrix} 1 \\ -1 \\ -2(b + a + 2t_0) \end{bmatrix}.$$

Thus the length of $\vec{v}_1$ is

$$\|\vec{v}_1\| = \|E_1(t_0) - E_1(0)\| = |t_0|\sqrt{1 + 1 + 4(b - a)^2}\,.$$

Noting that $E_2(0) = T_1(s_0)$, we see the length of $\vec{v}_2$ is

$$\|\vec{v}_2\| = \|T_1(s_0) + t_0 \begin{bmatrix} 1 \\ 1 \\ 2(b - a - 2s_0) \end{bmatrix} - E_2(0)\| = |t_0|\sqrt{1 + 1 + 4(b - a - 2s_0)^2}\,.$$

Thus the arclengths along E_1 and E_2 are related via

$$\frac{\|\vec{v}_2\|^2}{\|\vec{v}_1\|^2} = \frac{2+4(b-a-2s_0)^2}{2+4(b-a)^2}\,. \tag{A.3}$$

Note that this just depends on the parameter s_0. A similar calculation can be repeated for E_3. Assume the trisecants are of the form xyz where $x \in e_1$, $y \in e_2$ and $z \in e_3$. Suppose the first point x of the trisecant moves along edge e_1 with unit speed, then from Equation A.3 we know how the corresponding points y and z move along e_2 and e_3 respectively. Thus along the interval of trisecants the points move smoothly (and monotonically) along the corresponding edges.

Suppose now we have the doubly ruled surface which is a 1-sheeted hyperboloid of the form $x^2 + y^2 - z^2 = 1$ and assume the point (a, b, c) lies on this surface. Then a calculation shows that the vector equations of the two lines (one in each ruling) through (a, b, c) are

$$T_1(s) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + s \begin{bmatrix} b^2 - c^2 \\ -ab - c \\ -ac - b \end{bmatrix}, \qquad E_1(t) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + t \begin{bmatrix} b^2 - c^2 \\ -ab + c \\ -ac + b \end{bmatrix}.$$

We may repeat the previous calculations and again show that along the interval of trisecants the point move smoothly along the corresponding edges. These calculations are omitted as they are much more complex and don't give any new insights.

In fact we may do similar calculations for any doubly-ruled surface, but the computations become increasingly more complicated. It is more helpful to realize that any doubly-ruled surface may be obtained from either $z = y^2 - x^2$ or $x^2 + y^2 - z^2 = 1$ by a combination of rotations, translations and dilations. All of these maps send lines to lines. (In fact there is a (projective) transformation that maps a 1-sheeted hyperboloid to a hyperbolic paraboloid that again maps lines to lines.) Thus the computations we have for $z = y^2 - x^2$ hold for any other doubly-ruled surface. Hence along the interval of trisecants the points along the corresponding edges move smoothly and monotonically.

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Vita

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