ON THE TOTAL CURVATURE OF A KNOTTED SPACE CURVE.

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1. Introduction

1. M. Fenchel[1] has proved a theorem saying that a closed curve (in ordinary space) has total curvature $\geq 2\pi$. Recently, M. Borsuk gave a new proof of this theorem, holding for curves in n-dimensional space.[2] At the end of his paper, M. Borsuk posed the question of whether the total curvature of a nonplanar knotted curve is always $\geq 4\pi$.[3] The principal object of this paper is to give a positive response to this question (Theorem 3)[4]. Our proof is based on the simple but interesting fact that the total curvature of a knotted curve is equal to the average of those of its orthogonal projections (Theorem 2).

This remark permits the study of other such questions, like for example, that of the lower bound of the total curvature of curves belonging to a given knot type, etc. All these problems can perhaps be brought back to the questions about plane curves, but as these last have a rather combinatorial nature, we will not consider them here[5].

2. We will firstly indicate some usual definitions and we will introduce notation.

Let us imagine *closed curves* in ordinary space.[6] The rectangular coordinates of a point varying on such a curve C are given by a function of arc length s,[7]

$$C:\ r(s) = [x(s), y(s), z(s)] \quad (0 \leq s \leq l); \tag{1}$$

as the curve is closed, we always have $r(0) = r(l)$. We suppose that the imagined curve satisfies the following condition: [Translators note: piecewise C^2.]

a. The curve C admits a tangent (varying continuously) everywhere except for a finite number of points, which in addition correspond to the parameter values $0 \leq d_1 < \cdots < d_n < l$; $r''(s)$ is continuous in each closed interval $d_i \leq s \leq d_{i+1}$.

[1] W. Fenchel *Über Krümmung und Windung geschlossenser Raumkurven* (Math. Ann., vol. 101, 1929, p. 238-252).

[2] K.Borsuk, *Sur la courbure totale des courbes fermées* (Ann. Soc. Pol. Math., vol. XX, 1948, p. 251-265).

[3] Borsuk, *loc cit.*, p.265 (*Problème*); see also his note ([13]).

[4] In his *Problème*, M. Borsuk considers *regular closed* curves, that is to say nonplanar curves closed having a determined tangent at each point, which varies continuously with the point of contact. We have considered another class of curves (see *Introduction*, 2, property a) but the difference is not essential.

[5] We will return to this in another footnote, treating the mentioned questions plus more, the *global curvature* of a system of *linked* curves, etc.

[6] In the spaces of four or more dimensions, each curve homeomorphic to a circle is isotopic to a circle, this is why we will only consider ordinary space. Let us remark here that Theorems 1 and 2 are valid in n-dimensional space.

[7] This will always be possible for any rectifiable curve.

We make use of the following notions of the topology of closed curves. The *homeomorphic* curves C, C' are called *isotopic* if there exists a family of curves C_τ continuously dependent on τ $(0 \leq \tau \leq 1)$, C_τ is homeomorphic to C, $C_0 = C$ and $C_1 = C'$. *We say that C is a knot if it is homeomorphic to a circle,*[8] *but is not isotopic to a circle.* We will use the easily proved fact that if the curve C is a knot, then each of the polygons inscribed in C (and also having small sides) are also knots. [Translators note: since piecewise C^2.]

Let us consider three points on the curve C corresponding to the parameters a, b, c and let ρ_{abc} be the radius of the circle passing through these points. The curvature at a point s is defined by the limit (supposing it exists)[9]

$$\lim_{d\to 0} \frac{1}{\rho_{abc}} = \kappa(s) \quad (d = |a-s| + |b-s| + |c-s|). \tag{2}$$

We will always designate the angle between the vectors a, b by $\Phi(a, b)$, chosen so that

$$0 \leq \Phi(a, b) \leq \pi. \tag{3}$$

The following is another usual definition of curvature:

$$\kappa(s) = \lim_{|b-a|\to 0} \frac{\Phi[r'(a), r'(b)]}{|b-a|} = |r''(s)|. \tag{4}$$

We know that the limits (2) and (4) are equal, if $r''(s)$ is continuous. Finally we remark that we always have $\kappa(s) \geq 0$ (even for planar curves).

We will use the fact that the limits (2) and (4) are attained uniformly in s in each interval $s_1 \leq s \leq s_2$ or $r''(s)$ is continuous[10].

Let us move to *total curvature*. If $r(s)$ admits derivatives of second order and if the curve is regularly closed, that is to say if we have

$$r'(0) = r'(l) \quad [r(0) = r(l)],$$

then the total curvature $\chi(C)$ is defined by

$$\chi(C) = \int_0^l \kappa(s) ds. \tag{5}$$

Moreover, due to (4), we have

$$\chi(C) = \int |d\phi|, \tag{6}$$

where $d\phi$ is the infinitesimal variation of the angle of the tangent. On the other hand $|r'(s)| = 1$; $r'(s)$ is thus on the unit sphere, and $\chi(C)$ is the length of an arc of this first spherical image.[11]

[8] That is to say that a closed curve and without multiple points

[9] As for the definition of curvature, see for example, De La Vallée Poussin, *Cours D'Analyse infinitésimale* (8th edition), 1938, t. I, p. 280-289; Blaschke, *Vorlesungen über Differentialgeometrie* (3rd edition) Bd. I, 1930, p. 17-18 and 61-64.

[10] See for example, Pauc, *Les méthodes directes en géométrie différentielle*, Paris 1941, p. 124-125, theorems XXII and XXIII.

[11] This remark leads us to a more general definition of total curvature: if the curve $r'(s)$ is rectifiable and closed, we can define the total curvature of $r(s)$ as the length of the arc of $r'(s)$ [See Borsuk ([2]) p. 251-253]. After a theorem of Lebesgue, the curvature $\chi(s)[= |r''(s)|]$ of $r(s)$ exists in this case almost everywhere and the total curvature is given by formula (5). [Translators note: essentially C is $C^{1,1}$.]

This definition can be extended in a natural manner to more general curves than we have considered here, those having property a. It suffices to consider (6) as a Stieltjes integral, that is to say as the limit

$$\lim_{max|t_i-t_{i-1}|\to 0} \sum_{i=1}^{n} \Phi[r'(t_i), r'(t_{i-1})] = \int |d\phi| \quad (t_i \neq d_j). \tag{7}$$

We observe again that the total curvature is thus defined not only as the length of a curve traced out on the unit sphere and is made up for one part of the images corresponding to the continuous segments of $r'(s)$, and for the other part the arcs of great circles joining these arcs consecutively.

The total curvature of a closed polygon

$$P = A_1, \dots, A_n \ (= A_1A_2, \dots, A_{n-1}A_n, A_nA_1), \tag{8}$$

is easily calculated from (7)

$$\chi(P) = \sum_{i=1}^{n} \Phi(\overrightarrow{A_{i-1}A_i}, \overrightarrow{A_iA_{i+1}}) \tag{9}$$

[for closed polygons we suppose that $A_i = A_j$ for $i \equiv j \pmod{n}$].

We finally remark that after definition (7) for the total curvature of a curve C, we have the inequality

$$\chi(C) \geq 2\pi, \tag{10}$$

for closed planar curves; equality in (10) only holds for convex curves.

2. The total curvature of a curve, considered as a limit of those of its inscribed polygons.

Theorem 1. *Let P_r be a family of polygons inscribed in C $[C : r(s)]$. Suppose that the discontinuous points of $r'(s)$ figure amongst the vertices of all of the P_r and that the length of the longest side of P_r tends towards zero with $\frac{1}{r}$. Then the total curvature of P_r tends towards that of C.*

Proof: Imagine an arc C' of C, corresponding to the interval $d \leq s \leq d'$, where $r''(s)$ is continuous. Let $a, b, c\,(d \leq a < b < c \leq d')$ be three values of s and suppose

$$\theta = \Phi[r(b) - r(a), r(c) - r(b)],$$
$$\theta^* = \Phi\left[r'\left(\frac{a+b}{2}\right), r'\left(\frac{b+c}{2}\right)\right].$$

We will show that

$$\frac{\theta}{\theta^*} \to 1, \tag{11}$$

when $|c - a| \to 0$ and this is uniform over C'.

1. The continuity of $r'(s)$ means that the angle θ tends towards zero with $|c-a|$; we have thus for $|c - a|$ very small [Translators note: for any $\eta > 0$ we have the following,]

$$\sin\theta \leq \theta \leq (1 + \eta)\sin\theta.$$

2. We have indicated that the limit (4) holds uniformly in s (on the arc C' or where $r''(s)$ is continuous), thus we have

$$(1 - \eta)\kappa(b)\frac{c-a}{2} \leq \theta^* \leq (1 + \eta)\kappa(c)\frac{c-a}{2},$$

for $|c-a|$ small enough.

3. Let $\frac{1}{\kappa^*} = \rho_{abc}$ be the radius of the circle passing through the points $r(a), r(b), r(c)$ of C'; from (2) we have

$$(1-\eta)\kappa(b) \leq \kappa^* \leq (1+\eta)\kappa(b),$$

for $|c-a|$ sufficiently small.

4. Finally we have, for $|u-v|$ small enough,

$$1-\eta \leq \left|\frac{r(u)-r(v)}{u-v}\right| \leq 1+\eta,$$

as s is arclength, so that we have $|r'(s)| = 1$.

Let us now suppose that η is given, and choose $c-a$ small enough so that the inequalities $1-4$ are valid on the arc C'. (The last inequality for $u=a$, $v=b$, and $u=b$, $v=c$.)

Let the center of the circle passing through $r(a)$, $r(b)$, $r(c)$ be denoted by r^* and state

$$\theta' = \Phi\left[r(b)-r^*, \frac{r(a)+r(b)}{2} - r^*\right]$$
$$\theta'' = \Phi\left[r(c)-r^*, \frac{r(b)+r(c)}{2} - r^*\right]$$
$$(\theta' + \theta'' = \theta; \quad \theta', \theta'' > 0).$$

We have

$$\frac{\sin\theta'}{\kappa^*} = \left|\frac{r(b)-r(a)}{2}\right|, \quad \frac{\sin\theta''}{\kappa^*} = \left|\frac{r(c)-r(b)}{2}\right|.$$

Following 1. and 2. we have

$$\frac{\theta}{\theta^*} = \frac{\theta'+\theta''}{\theta^*} \leq \frac{(1+\eta)(\sin\theta' + \sin\theta'')}{(1-\eta)\kappa(b)\frac{c-a}{2}} = A.$$

By applying the inequalities in 3. we obtain

$$A \leq \frac{(1+\eta)^2\kappa(b)\left(\left|\frac{r(b)-r(a)}{2}\right| + \left|\frac{r(c)-r(b)}{2}\right|\right)}{(1-\eta)\kappa(b)\left(\frac{b-a}{2} + \frac{c-b}{2}\right)},$$

from which, when using 4. and the fact that $\frac{p}{q} \leq \frac{p'}{q'}$ implies

$$\frac{p}{q} \leq \frac{p+p'}{q+q'} \leq \frac{p'}{q'},$$

we derive the inequality

$$\frac{\theta}{\theta^*} \leq \frac{(1+\eta)^3}{1-\eta}.$$

An analogous calculation gives

$$\frac{\theta}{\theta^*} \geq \frac{(1+\eta)^3}{1-\eta},$$

and from these two last inequalities we obtain (11).

Suppose firstly that $r''(s)$ is continuous everywhere. In order to compare the total curvature of a polygon P_r with that of C, we designate by A_{r_i} the vertices and by

$$\theta_{r_i} = \Phi(\overrightarrow{A_{r_{i-1}}A_{r_i}}, \overrightarrow{A_{r_i}A_{r_{i+1}}}) \quad (i = 1, \ldots, n_r),$$

the exterior angles of which the sum is $= \chi(P_r)$. From (10), we have for large enough r

$$(1-\epsilon)\sum_{i=1}^{n_r}\theta^*_{r_i} \leq \sum_{i=1}^{n_r}\frac{\theta_{r_i}}{\theta^*_{r_i}}\theta^*_{r_i} \leq (1+\epsilon)\sum_{i=1}^{n_r}\theta^*_{r_i}$$

where we denote by $\theta^*_{r_1}$ the angle of the tangents (to C) in the middle of the consecutive arcs $A_{r_{i-1}}A_{r_i}, A_{r_i}A_{r_{i+1}}$ ($\epsilon > 0$ is given in advance). But from (7)

$$\sum_{i=1}^{n_r}\theta^*_{r_i},$$

is a sum of approximation of $\chi(C)$, which completes the proof in this particular case.

As for the general case, it suffices to subdivide C into arcs C_k partitioned by the points of discontinuity of $r'(s)$. The ends of the C_k figure among the vertices of P_r and these decompose into partial polygons. One sees that the total curvature of these polygons tends towards that of C_k, the angle of the sides separated by the extremities of C_k tend towards the angles of the left and right tangents of these points.

Let us finally remark that, if the total curvature of the polygons is defined by formula (9), then the total curvature of a general curve C could be defined by those of polygons inscribed inside C, in the same manner as the length of a curve is defined by that of inscribed polygons. Theorem 1 shows that this definition agrees with the classical definition for quite a large class of curves[12].

3. The total curvature of a curve considered as the average of those of its orthogonal projections.

Lemma 1. *Let a_n,b_n be the orthogonal projections of the vectors a, b on a plane whose normal vector is n. Suppose that $\Phi^*(n; a, b) = \Phi(a_n, b_n)$. We have*

$$\theta = \Phi(a,b) = \frac{1}{4\pi}\iint_S \Phi^*(n; a, b)\, d\omega, \tag{12}$$

where $d\omega$ denotes an area element of the unit sphere S.

Proof. Let us firstly show that the integral shown in (12) depends only on the angle of the vectors a and b.

For $b = \lambda a$ ($\lambda > 0$) the two sides of (12) are zero.

For $b = -\lambda a$ ($\lambda > 0$) we have $\Phi^*(n; a, b) = \pi$ (except for $n = \pm\mu a$, which does not change the integral.) We have

$$\pi = \frac{1}{4\pi}\iint_S \Phi^*(n; a, -\lambda a)\, d\omega, \quad (\lambda > 0).$$

[12]See Blaschke ([9]) p. 17-18.

In the general case, where the vectors a, b are linearly independent, let n_0 be normal to their plane. Let a', b' be arbitrary vectors subject only to the condition

$$\Phi(a', b') = \Phi(a, b).$$

Let us bring $\frac{a}{|a|}$ to $\frac{a'}{|a'|}$ and $\frac{b}{|b|}$ to $\frac{b'}{|b'|}$ by a rotation around the origin (the base point of the vectors considered.) In transforming integral (12) by this rotation, one sees that (12) depends only on the angle of the vectors a, b, that is to say

$$\frac{1}{4\pi}\iint_S \Phi^*(n; a, b)\, d\omega = f(\theta), \quad \theta = \Phi(a, b). \tag{13}$$

Let us show that the function $f(\theta)$ is a solution of the functional equation

$$f(\theta_1 + \theta_2) = f(\theta_1) + f(\theta_2) \quad (0 \leq \theta_1 + \theta_2 \leq \pi;\ \theta_i > 0). \tag{14}$$

Consider three coplanar vectors a, b, c chosen in a way so that

$$\theta_1 = \Phi(a, b), \quad \theta_2 = \Phi(b, c), \quad \theta_1 + \theta_2 = \Phi(a, c).$$

We have then

$$\Phi^*(n; a, c) = \Phi^*(n; a, b) + \Phi^*(n; b, c),$$

and integration over the unit sphere gives us (14). Moreover, we have $0 \leq f(\theta) \leq \pi$ $(0 \leq \theta \leq \pi)$; from (13) $f(\theta) = \pi$ thus $f(\theta) = \theta$ $(0 \leq \theta \leq \pi)$. [13]

Theorem 2. *Let C_n be the orthogonal projection of a curve C on the plane normal to n, $\chi(C)$, $\chi(C_n)$ the total curvatures of C and C_n. If $\chi(C_n) \leq \kappa$ (independent of n), then we have*

$$\chi(C) = \frac{1}{4\pi}\iint_S \chi(C_n)\, d\omega\,, \tag{15}$$

where the integral counts all directions, that is to say, all the unit sphere S.

Proof. This theorem is a consequence of Theorem 1 and of Lemma 1. For the proof, chose a family of polygons P_r inscribed in C for which we have

$$\lim_{r\to\infty} \chi(P_r) = \chi(C) \tag{16}$$

This means that

$$\lim_{r\to\infty} \chi(P_{rn}) = \chi(C_n) \tag{17}$$

where P_{rn} denotes the projection of P_r on a plane having normal n.

From (17), we have

$$\mathcal{T} = \frac{1}{4\pi}\iint_S \chi(C_n)\, d\omega = \frac{1}{4\pi}\iint_S \lim_{r\to\infty} \chi(P_{rn})\, d\omega,$$

and from a theorem of Lebesgue

$$\mathcal{T} = \lim_{r\to\infty} \frac{1}{4\pi}\iint_S \chi(P_{rn})\, d\omega, = \lim_{r\to\infty} \chi(P_r) = \chi(C),$$

which completes the proof. [14]

[13] Theorem 1 can be generalized, but we do not treat this generalization here. We will return to these questions in another footnote. [Translators note: possibly Lemma 1 is meant here.]

[14] As for the hypothesis $\chi(C_n) \leq \kappa$ of Theorem 2, we remark that there are curves for which $\chi(C) < \infty$ and $\chi(C') = \infty$ simultaneously, where C' is a projection of C.

4. On the total curvature of a nonplanar closed curve

By using formula (15) we can prove

Theorem. *(Fenchel) Let C be a nonplanar closed curve and denote its total curvature by $\chi(C)$. We then have the inequality $\chi(C) \geq 2\pi$, where equality only holds for convex planar curves.*

Proof. First, let us prove

Proposition 1. *If $\chi(C) = 2\pi$, all the projections of C are convex curves (thus having total curvature of 2π).*

Let us suppose that the proposition is not true, and let C_{n_0} be a projection of C (on a plane whose normal vector is n_0), which is not convex. There is thus a neighborhood of n_0 such that C_n is not convex for each n belonging to this neighborhood (recall that the limit of convex curves is a convex curve). One then has the inequality

$$\chi(C_n) > 2\pi$$

in this neighborhood of n_0. From (10) and the inequality above, formula (15) furnishes the inequality $\chi(C) > 2\pi$ contrary to our hypothesis. The proposition is thus proved.

Let us now prove Fenchel's theorem. The first proposition of the theorem is an immediate consequence of (10) and (15), we have only to consider the case $\chi(C) = 2\pi$. By using the proposition we will prove in this case that C is a convex planar curve.

Let P, Q and R be any three points of the curve C, S a point on the segment PR. Consider the projection C' of the curve C on the plane whose normal vector is $\overrightarrow{SQ}$. Denote by P', Q', R', $S'(= Q')$ the projection of P, Q, R, S respectively. From the proposition, C' is a convex curve. The points P', R', Q', are collinear, that is to say, C' contains the segment $P'Q'$. Then the arc P, Q, R of C is contained in a plane. Similarly this is true for each group of three points of the curve C, so C is also contained in a plane. Now from (10), C is a convex curve, so the proof of the theorem is completed.

5. On the inequality $\chi(C) \geq 4\pi$ for knots C.

In the first part of this paper we considered the projections of a closed polygon without multiple points from the combinatorial point of view, and we gave a (necessary) criterion allowing one to recognize when the polygon is a knot.

In that which follows, let P be a closed polygon without multiple points. A projection P' of P is said to be regular if any line of a projection meets at most two sides of P.

Proposition 2. *Let P be a knotted polygon, P' one of its regular projections on the plane Π. There exists a point O on the plane Π such that all the rays issuing from O and belonging to the plane Π cut P' in $k \geq 2$ different points (or in a multiple point).*

Proof. As P' is a regular projection, its double points belong to two of its sides, they are not vertices and are situated on the boundary of three or four regions. The polygon P' cuts the plane into regions with the following properties:

a. These regions separate into two classes $\mathcal{U}$, $\mathcal{V}$ such that a region $U \in \mathcal{U}$ and a region $V \in \mathcal{V}$ don't have common sides[15]. [Translators note: This is possibly a mistake, Fary is describing black and white colouring for a bipartite graph. Try instead: $\mathcal{U}$ touches only $\mathcal{V}_i$'s.]

Let U_0 be the region which contains the point at infinity, and let $U_0 \in \mathcal{U}$. Denote by $V_1, \ldots, V_{i_1}$ the regions which are next to U_0 and let $V_i \in \mathcal{V}$ $(1 \leq i \leq i_1)$. Let $U_1, \ldots, U_{j_1}$ be the regions which are adjacent to at least one of the $V_i (1 \leq i \leq i_1)$, etc. Let us show that each domain belongs to one of these classes $\mathcal{U}, \mathcal{V}$: in the contrary case, we have a closed polygonal loop L which has an odd number of points in common with P'. Deform L, the number of these intersection points with P' varies, but stays odd; when L is contracted to a single point, the number is nevertheless zero. This contradiction proves our proposition *a*.

b. If P' is a regular projection of a knot, there is a region not adjacent to the region containing the point at infinity. (We thus have one $U_1 \neq U_0$).

Suppose to the contrary that all the regions different from U_0 have a side in common with it, that is to say they are adjacent. Let us show that in this case one can deform P into a polygon P_1 having a projection P'_1 on the plane Π, which cuts Π into a number of smaller regions, each adjacent to the region containing the point at infinity.

From the region V_1 of P' we go to a region V_{k_1} which shares a vertex (double point of P') with V_1; the region $V_{k_{r+1}}$ will be defined in the same manner as V_{k_r} (the construction is naturally not unique). If $V_{k_m} = V_{k_n}$ one can construct a piecewise linear closed line, whose interior (using *a*) contains a region that the other regions separate from the one which contains the point at infinity. This case being excluded, one arrives at a region V_{k_s} for which $V_{k_{s+1}}$ does not exist, that is to say such that its boundary is a *loop* having only one double point of P'. By a suitable deformation one can remove the *loop* from the projection of P'.

One can thus deform P into a polygon $\bar{P}$, for which the projection $\bar{P}'$ has only one finite region, $\bar{P}$ is thus isotopic to a circle. *b* is thus completely proved.

Let us choose the point O in the interior of $U_1 (\neq U_0)$. One sees immediately that O has the desired property: all rays issuing from O cut P' in $k \geq 2$ different points (or in a multiple point). The proof of the proposition is thus completed.

Theorem 3. *Each knot C has total curvature $\chi(C) \geq 4\pi$.*

Proof: Let P' be a regular projection. We claim that

$$\chi(P') \geq 4\pi. \tag{18}$$

Let $P' = A_1, \ldots, A_n$. Let us choose the point O using the property of the proposition and the fact that the points $OA_sA_t (s \neq t; 1 \leq s, t \leq n)$ are not collinear. Let us suppose

$$\gamma_k = \Phi(\overrightarrow{OA_k}, \overrightarrow{OA_{k+1}})$$
$$\theta_k = \Phi(\overrightarrow{A_{k-1}A_k}, \overrightarrow{A_kA_{k+1}}) \quad [A_i = A_j, \text{if } i \equiv j (\operatorname{mod} n)]$$
$$\alpha_k = \Phi(\overrightarrow{A_kO}, \overrightarrow{A_kA_{k+1}}).$$

From the definition of O, we have the inequality

$$\sum_{k=1}^{n} \gamma_k \geq 4\pi. \tag{19}$$

[15]C. Bankwitz *Über die Torsionszahlen der alternierenden Knoten* (Math. Ann., t. 103, 1930, p. 145-146).

We will prove that

$$\theta_k \geq \alpha_{k-1} + \gamma_{k-1} - \alpha_k \quad (k = 1, \ldots, n). \tag{20}$$

The lines OA_k and $A_{k-1}A_k$ cut the plane into four angular regions (The lines are different according to the choice of O). Designate by I the region containing the triangle $OA_{k-1}A_k$ (which determines the positive sense of the plane), and let II, III, IV be the other domains taken in the positive sense. For the value of θ_k we obtain (see figures I, II, III, and IV) $\theta_k = \alpha_{k-1} + \gamma_{k-1} + \alpha_k$ if A_{k+1} is in I, $\theta_k = 2\pi - (\alpha_{k-1} + \gamma_{k-1} + \alpha_k)$ if A_{k+1} is in II, $\theta_k = -\alpha_{k-1} - \gamma_{k-1} + \alpha_k$ if A_{k+1} is in III, $\theta_k = \alpha_{k-1} + \gamma_{k-1} - \alpha_k$ if A_{k+1} is in IV.

Inequality (20) is thus immediate. In taking the sum, we obtain

$$\sum_{k=1}^{n} \theta_k \geq \sum_{k=1}^{n} \gamma_k$$

and from (19), inequality (18).

Let C now be a given knot. If $\epsilon > 0$, one can determine a polygon inscribed in C and making a knot such that

$$\chi(C) + \epsilon \geq \chi(P).$$

For all regular projections P_n of P we have $\chi(P_n) \geq 4\pi$; the singular directions are situated in planes and hyperboloids of finite number and thus have a zero measure on the surface of the unit sphere. An integration on the surface of the sphere of the two members of (18) gives

$$\chi(C) + \epsilon \geq 4\pi,$$

and as ϵ is arbitrary (> 0) we obtain theorem 3.

Let us finally remark that one can construct knots (belonging to the simplest topological class) with total curvature $\leq 4\pi + \epsilon$ ($\epsilon > 0$ arbitrary); that is to say that the inequality of Theorem 3 is the best possible.

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Addition to the correction of the proofs (31st May 1949). I have received a letter from Mr Borsuk which tells me that Theorem 3 has been previously proved by Mr H. Hopf. He uses the theorem of Miss Pannwitz which affirms that for all knots one can find a line cutting in at least four points.

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