Knots and Minimum Distance Energy

Special Studies, Department of Mathematics and Statistics, Smith College, Spring 2008
Rosanna Speller, Advisor: Elizabeth Denne

Abstract

Professor Elizabeth Denne and I continue work I started in a research program (summer 2007). We aim to find which polygonal knots have least Minimum Distance Energy. I previously showed that the energy is minimized for convex polygons. We hope relating the energy to chords of polygons will be a helpful step towards showing that regular n-gons have the least minimum distance energy for all polygonal knots.

Preliminary Definitions and Theorems

"Simon's Minimum Distance Energy" [3, 5] Ror a pair of nonconsecutive edges, X and Y, of an n-gon is $U_{md}(X,Y) = \frac{\ell(X)\ell(Y)}{md(X,Y)^2}$. Here, $\ell(X)$ is the length of X and $md(X,Y)$ is the minimum distance between X and Y. The Minimum Distance Energy of a polygon, P, is

$$U'_{md}(P) = \sum_{\text{all edges } X} \; \sum_{Y \neq X \text{nor adjacent}} U_{md}(X,Y).$$

The **convex hull** of an n-gon, P, is the smallest convex set containing all vertices of P and is denoted $H(P)$. The boundary is denoted $\partial H(P)$.

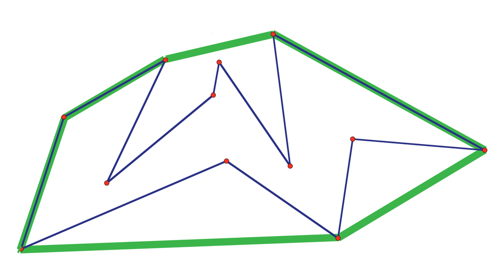

Figure 1: The convex hull (green) of a non-convex polygon (blue)

A **pocket** is a set of edges of a polygon not in $\partial H(P)$ between the vertices i and j on $\partial H(P)$. Its **pocket lid** is the line segment $\overline{ij}$.

A **flip** is the reflection of a pocket across a pocket lid.

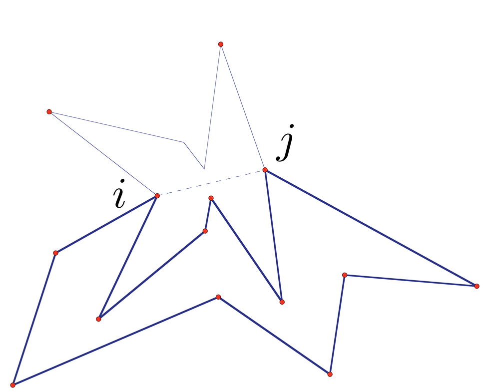

Figure 2: Flipping a pocket over its pocket lid, $\overline{ij}$

Erdös-Nagy Theorem. [1, 6] Every simple planar polygon can be made convex with a finite number of flips.

A **stretch** is made by a change in angles. For P and P', polygons with corresponding lengths, P' is a **stretched** version of P, if $\forall\ x,\ y\ \in P$ and corresponding $x',\ y'\ \in P'n$, $|x-y| \leq |x'-y'|$ [4].

Sallee's Lemma. [4] If P is a non-convex polygon in $\mathbb{E}^3$, $\exists$ a stretched polygon P', which is planar and convex, such that $\forall$, x, $y \in P$, with x and y not on the same edge, and for corresponding x', $y' \in P'$, $|x-y| < |x'-y'|$.

Previous Results[1]

Theorem. If P is a planar n-gon with minimized U'_{md}, then P is convex.

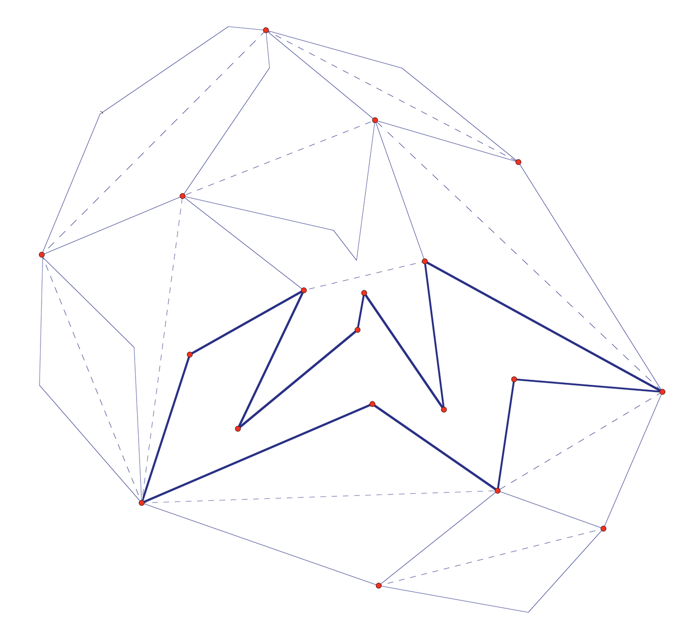

Figure 3: Convex polygon made by flipping

Theorem. If P is a polygon in $\mathbb{E}^3$ there exists a convex planar polygon, P', created by stretching such that $U'_{md}(P) \geq U'_{md}(P')$.

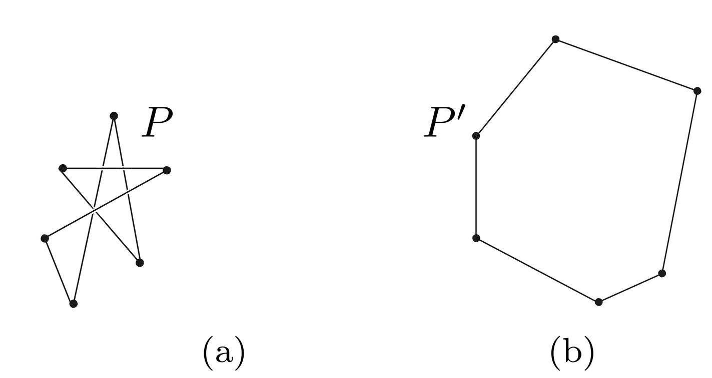

Figure 4: (b) A convex polygon made by stretching the polygon in (a)

Regular n-gons, R_n.

The minimum distance energy of R_n when n is odd is:

$$U'_{md}(R_n) = 2n \cdot \sin^2\left(\frac{\pi}{n}\right) \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{1}{\sin^2\left(\frac{j\pi}{n}\right)}$$

When n is even:

$$U'_{md}(R_n) = n \cdot \sin^2\left(\frac{\pi}{n}\right) \left(\frac{1}{\sin^2\left(\frac{\pi(n-2)}{2n}\right)} + 2 \cdot \sum_{j=1}^{\frac{n}{2}-2} \frac{1}{\sin^2\left(\frac{j\pi}{n}\right)} \right)$$

New Investigations

Conjecture. Regular n-gons have least minimum distance energy.

Lükő's Theorem II. [2] Let the vertices of a n-gon be labeled $1, 2, \ldots, n$ and let $r_{i,l}$ denote the distance between vertices i and $i+l$. Let a be a constant greater than or equal to the length of every edge (denoted $r_{i,1}$) of the n-gon. Let $g(t)$ be an increasing, concave function, then,

$$\frac{1}{n} \sum_{i=1}^{n} g(r_{i,l}^2) \leq g\left(a^2 \frac{\sin^2(\frac{l\pi}{n})}{\sin^2(\frac{\pi}{n})}\right)$$

for all $n \geq 4$, with equality if and only if the n-gon is regular.

When $g(t) = t$, this theorem implies the average squared distance between the vertices of an n-gon is maximized by the regular n-gon.

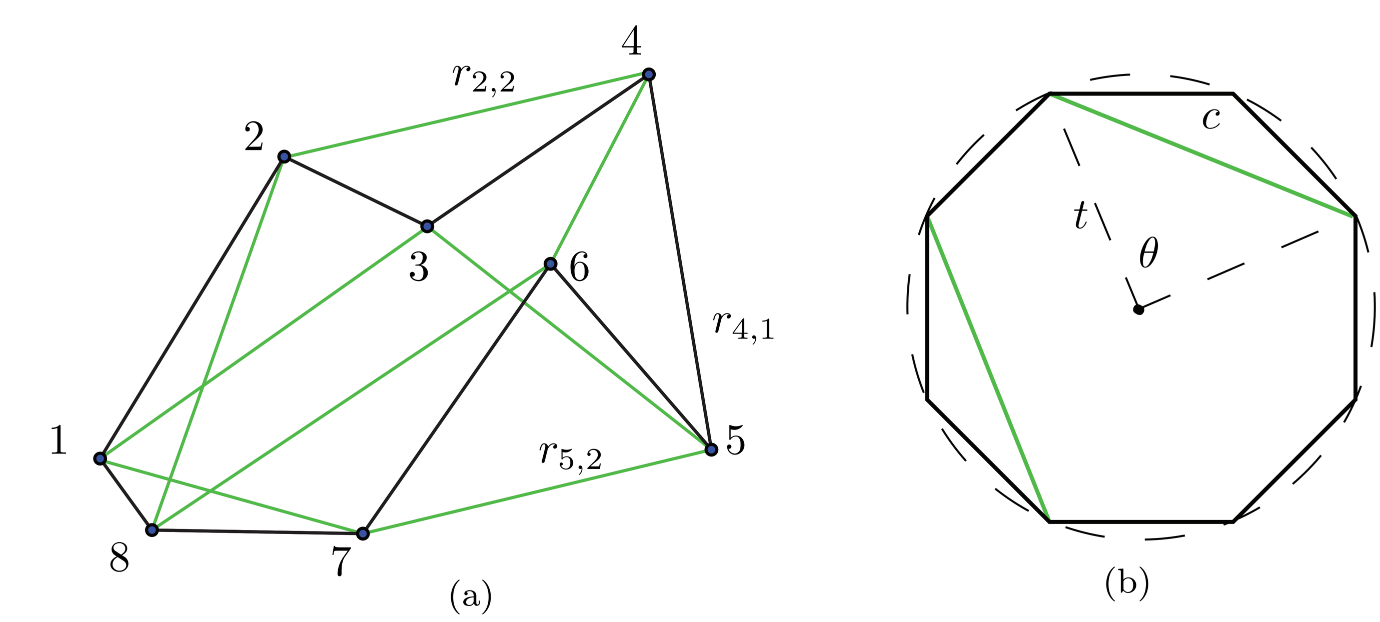

Figure 5: (a) Polygon with edges ($r_{i,1}$) in black, each $r_{i,2}$ is given in green. (b) Regular octagon inscribed in circle of radius t; distances between vertices are the length of a chord $\ell(c) = 2t\sin(\frac{\theta}{2})$.

References

[1] P. Erdös, Problem number 3763, Amer. Math. Monthly **42** (1935) 627.

[2] G. Lükő, On the mean length of the chords of a closed curve, **4** (1966) 23-32.

[3] E. J. Rawdon and J. K. Simon, Polygonal approximation and energy of smooth knots, J. Knot Theory Ramifications **15**:4 (2006), 429-451.

[4] G. T. Sallee, Stretching chords of space curves, Geometriae Dedicata **2** (1973) 311-315.

[5] J. Simon, Energy functions for polygonal knots, J. Knot Theory Ramifications **3**:3 (1994), 299-320.

[6] G. Toussaint, The Erdös-Nagy theorem and its ramifications, Computational Geometry **31** (2005) 219-236.

[1] R.S. thanks adviser Dr. R. Trapp of California State University, San Bernardino and 2007 REU program, jointly sponsored by CSUSB and NSF-REU Grant DMS-0453605